Standard Koszul self-injective special biserial algebras

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Abstract

The aim of this paper is to establish a connection between the standard Koszul and the quasi-Koszul property in the class of self-injective special biserial algebras. Furthermore, we give a characterization of standard Koszul symmetric special biserial algebras in terms of quivers and relations.

1 Introduction

The concept of biserial algebras was introduced by Tachikawa in [12], while the significant subclass of special biserial (SB) algebras was first studied in the work of Skowroński and Waschbüsch [11]. Self-injective, in particular, symmetric special biserial (SSB) algebras appear in the theory of modular representations of finite groups (see [9] and [10]), and play an important role in complex representations of the Lorentz group [5].

In this paper we study the quasi-Koszul and standard Koszul properties for self-injective SB algebras. Ágoston, Dlab and Lukács showed in [2] that every quasi-hereditary standard Koszul algebra is a quasi-Koszul algebra. Though the standard Koszul property is naturally considered for quasi-hereditary or at least standardly stratified algebras, it may still be interesting to examine standard Koszul algebras in general, along with the question whether standard Koszul algebras are all quasi-Koszul in a given class. We focus mainly on this implication.

In the process, we give a full description of the (minimal) projective resolutions of simple modules over self-injective SB algebras that have no uniserial projective modules (Section 2). Here we use a path-building technique based on the work of Antipov and Generalov [3] on SSB algebras, which is similar to the one that was used for monomial algebras (cf. [6]).

Section 3 gives a description of standard Koszul SSB algebras in terms of quivers and relations.

Throughout this paper, K is an arbitrary field, and A denotes a finite-dimensional basic connected K-algebra with a fixed complete ordered set of primitive orthogonal idempotents $\{e_1, \dots, e_n\}$. Modules are right modules, unless otherwise stated. The indecomposable projective summands of the canonical decomposition of the regular module

$$A_A = e_1 A \oplus \ldots \oplus e_n A$$

are sometimes denoted by $P(i)$, and their simple top by $S(i)$. The corresponding left modules are $P^\circ(i)$ and $S^\circ(i)$.

Let $\varepsilon_i = e_i + e_{i+1} + \ldots + e_n$, and $\varepsilon_{n+1} = 0$ for convenience. With this notation, the standard and proper standard modules are defined as $\Delta(i) = e_i A / e_i A \varepsilon_{i+1} A$ and $\overline{\Delta}(i) = e_i A / e_i (\mathrm{rad}\ A) \varepsilon_i A$, respectively. One can define the standard and proper standard left modules $\Delta^\circ(i)$ and $\overline{\Delta}^\circ(i)$ analogously.

Let $M \in$ mod-A have the minimal projective resolution

$$\ldots \xrightarrow{\alpha_{k+1}} P_k \xrightarrow{\alpha_k} \ldots \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \to 0,$$

and let $\Omega_k(M) = \ker \alpha_{k-1}$ be the kth syzygy of M. (If $M = S(i)$, then we write $P_k(i)$ for the kth term and $\Omega_k(i)$ for the kth syzygy.) Usually, if there is no risk of misunderstanding, we omit the argument of Ω_k.

Let M be a submodule of N. We say that M is a *top* submodule of N ($M \overset{t}{\leq} N$) if $(\mathrm{rad}\ N) \cap M = \mathrm{rad}\ M$. Using the concept of *top* submodules, we can introduce the subclasses $\mathcal{C}_A^i$ $(i \geq 0)$ of mod-A and also the notion of top resolutions, which generalizes linear resolutions for the non-graded case. A module M is in $\mathcal{C}_A^i$ if $\Omega_j \overset{t}{\leq} \mathrm{rad}\ P_{j-1}$ holds for all $j \leq i$, and let $\mathcal{C}_A^0 =$ mod-A. The module M is a *quasi-Koszul module* if $M \in \mathcal{C}_A := \cap_{i=0}^{\infty} \mathcal{C}_A^i$, i.e. if M has a *top resolution*. An *algebra* A is *quasi-Koszul* if all of its simple right modules are quasi-Koszul, while A is *standard Koszul* if $\Delta(i) \in \mathcal{C}_A$ and $\overline{\Delta}^\circ(i) \in \mathcal{C}_{A^\circ}$ for all i.

2 Self-injective special biserial algebras

Let Γ be a quiver and I an admissible ideal of $K\Gamma$. We write the product of two arrows $\alpha : i \to j$ and $\beta : j \to k$ as $\alpha\beta : i \to j \to k$. An algebra $A \cong K\Gamma/I$ is said to be *special biserial*, or SB for short, if for each vertex v of Γ, there are at most two arrows starting, and at most two arrows ending at v, furthermore, for each arrow α there exists at most one arrow β and at most one arrow γ such that $\beta\alpha, \alpha\gamma \notin I$.

An algebra A is *self-injective* if A_A is an injective A-module. A is a *Frobenius algebra* if $A_A \cong \mathrm{Hom}_K({}_A A, K)$ as right modules. Frobenius algebras are always self-injective, on the other hand, every self-injective basic algebra is Frobenius [4]. If A is a Frobenius algebra, then there exists a linear function $\varphi : A \to K$ such that $\ker \varphi$ does not contain any nontrivial right or left ideal of A. We call a Frobenius algebra *symmetric* if the above *Frobenius function* is symmetric, i.e. $\varphi(ab) = \varphi(ba)$ for all $a, b \in A$. By an *SSB algebra*, we mean an SB algebra with such a fixed *symmetric form* φ.

Similarly to the ideas of Antipov and Generalov in [3] about SSB algebras, we introduce a function δ, which operates on the (scalar multiples of) paths of a self-injective SB algebra A. Since A is self-injective, soc $P(i)$ and soc $P^\circ(i)$ are simple modules for all i. So for every path $u : i \rightsquigarrow j$, there exist paths v, w such that uv, wu are maximal nonzero paths, which generate soc $P(i)$ and soc $P^{\circ}(j)$, respectively. The function φ cannot vanish on the socle of any indecomposable projective summand, so there is a unique scalar multiple $\delta(u)$ of v such that $\varphi(u\delta(u)) = 1$; and this definition extends naturally to nonzero scalar multiples of paths. Note that in the symmetric case $v = w$, and uv is always a cycle [3].

Let $A = K\Gamma/I$ denote a self-injective SB algebra. There are two types of indecomposable projective modules over A. The module e_iA can be either *uniserial* (i. e. its submodules form a chain), or by the definition of biserial algebras (cf. [11]) and self-injectivity, its radical can be written as a sum $U + V$ of two uniserial submodules U and V such that $U \cap V = \text{soc } e_iA$ is a simple module. Moreover, we see that each of the modules U and V is generated by an arrow, that is, $U = \alpha A$ and $V = \beta A$, where α and β are the two distinct arrows starting at the vertex i in the quiver Γ.

We are going to prove that, apart from a few trivial exceptions, the indecomposable projective modules of a quasi-Koszul or standard Koszul self-injective SB algebra cannot be uniserial, and we construct the (minimal) projective resolutions of simple modules in these cases.

In the lemmas and propositions of Section 2, we always assume that $A = K\Gamma/I$ is a self-injective connected SB algebra.

Lemma 2.1. *Let αA be the submodule of e_iA generated by the arrow α starting at the vertex i. Then αA is a top submodule of* rad e_iA. *Moreover, if $\alpha A \leq M \overset{t}{\leq} \text{rad } e_iA$, then M is either αA or* rad e_iA.

Proof. If e_iA is uniserial, then the statement clearly holds. Suppose that e_iA is not uniserial, and let β be the other arrow starting at i. Then

$$\text{rad}^2\ e_iA \cap \alpha A = \text{rad}\,(\alpha A + \beta A) \cap \alpha A = \text{rad}\ \alpha A + \underbrace{\text{rad}\ \beta A \cap \alpha A}_{\text{soc}\ e_iA} = \text{rad}\ \alpha A.$$

So $\alpha A \overset{t}{\leq} \text{rad } e_iA$. Let $M \geq \alpha A$ be a top submodule of rad e_iA. Using Lemma 1.1 of [1], we get

$$\tilde{M} := M/\alpha A \overset{t}{\leq} \text{rad}\ e_iA/\alpha A,$$

but since rad $e_iA = \alpha A + \beta A$ and $\alpha A \cap \beta A = \text{soc } e_iA$,

$$\tilde{M} \overset{t}{\leq} \text{rad}\ e_iA/\alpha A \cong \beta A/\,\text{soc}\ \beta A.$$

The module $\beta A/\,\text{soc } \beta A$ is uniserial, so if $\tilde{M}$ is a top submodule of rad $e_iA/\alpha A$, then $\tilde{M} = 0$ or $\tilde{M} = \text{rad } e_iA/\alpha A$, that is, $M = \alpha A$ or $M = \text{rad } e_iA$. □

Lemma 2.2. *Suppose that e_iA is an at least three-dimensional uniserial module for some i. If its simple top $S(i)$ is in $\mathcal{C}_A^m$, then $\Omega_k(i)$ is also uniserial for all $1 \leq k \leq m$, moreover, for all such k, the syzygy $\Omega_k(i)$ is generated by an arrow, and* $\dim_{\mathrm{K}}\ \Omega_k(i) \geq 2$.

Proof. We prove the lemma by induction on k. The first step is trivial. For the induction step, we need the following. Take a submodule $U = \alpha A$ generated by the arrow $\alpha : j \to \ell$ in the indecomposable projective module $e_j A$ such that $U \in \mathcal{C}_A^1$ and $\dim_{\mathrm{K}} U \geq 2$. Let $\varphi : \mathcal{P}(U) \to U$ be its projective cover. U is uniserial by the SB property but $\mathcal{P}(U) = e_\ell A$ is not, since otherwise $\ker \varphi \leq \operatorname{rad}^2 \mathcal{P}(U)$, contradicting $U \in \mathcal{C}_A^1$. By the SB property, $\alpha\beta = 0$ for one arrow β starting at ℓ. Thus $\beta A \leq \ker \varphi \overset{t}{\leq} \operatorname{rad} e_\ell A$, so Lemma 2.1 gives that $\ker \varphi$ is either βA or $\operatorname{rad} e_\ell A$. But the latter is impossible, since $\dim_{\mathrm{K}} U \geq 2$. Finally, βA is at least two-dimensional, since $\mathcal{P}(U)$ is not uniserial. □

Lemma 2.3. *Suppose that $e_i A$ is an at least three-dimensional uniserial module for some i. Then $S(i) \notin \mathcal{C}_A$.*

Proof. First, observe that if $X, Y \in \text{mod-}A$ and

$$0 \to Y \to P_m \to \ldots \to P_0 \to X \to 0$$

is a non-split exact sequence with indecomposable projective intermediate terms, then we have $\Omega_{m+1}(X) \cong Y$ and by the self-injectivity $\Omega^{-(m+1)}(Y) \cong X$, where $\Omega^{-k}(Y)$ is the kth cosyzygy of the module Y.

Now, assume that $S(i) \in \mathcal{C}_A$, consequently, $\Omega_1(i) = \operatorname{rad} P(i) \in \mathcal{C}_A$. Lemma 2.2 yields that the submodule $\Omega_k(i)$ is generated by an arrow in $P_{k-1}(i)$ for each k, moreover, $\dim_{\mathrm{K}} \Omega_k(i) \geq 2$. This also implies that every projective term in the projective resolution of $S(i)$ is indecomposable. But there exist only finitely many modules of the form αA, so there exists a smallest index k and a smallest integer h for which $\Omega_k(i) \cong \Omega_{k+h}(i)$. By our previous observation,

$$S(i) \cong \Omega^{-k}(\Omega_k(i)) \cong \Omega^{-k}(\Omega_{k+h}(i)) \cong \Omega_h(i),$$

which contradicts $\dim_{\mathrm{K}} \Omega_h(i) \geq 2$. □

We have seen in Lemma 2.3 that $e_i A$ cannot be uniserial if $\dim_{\mathrm{K}} e_i A \geq 3$ and A is quasi-Koszul. However, there are cases when $e_i A$ is uniserial. If A is local, then A_A may be uniserial: K and $K[x]/\langle x^2 \rangle$ are both quasi-Koszul and standard Koszul algebras. If A is non-local and there exists at least one index i for which $e_i A$ is uniserial, then the structure of the algebra is very special as we will see in the next proposition.

Proposition 2.4. *Suppose that A is non-local and quasi-Koszul. Assume that there exists an index i for which the module $e_i A$ is uniserial. Then all the indecomposable projective modules are uniserial. Furthermore, $A \cong K\Gamma/I$, where Γ is a directed cycle on n vertices, and I is generated by all the paths of length 2.*

Proof. First, observe that if $e_i A$ is uniserial for some i, then $\dim_{\mathrm{K}} e_i A = 2$ by Lemma 2.3 and the connectedness of A. Suppose that there exists a non-uniserial indecomposable projective module. Let $\mathcal{U} = \{e_i A \mid e_i A \text{ is uniserial}\}$ and $\mathcal{N} = \{e_j A \mid e_j A \text{ is not uniserial}\}$. The elements of $\mathcal{U}$ are two-dimensional projective-injective modules, so the image of every non-trivial morphism between elements of $\mathcal{U}$ and $\mathcal{N}$ (in either direction) is simple. Let us consider the bijection $f : \mathcal{U} \cup \mathcal{N} \to \mathcal{U} \cup \mathcal{N}$ which maps each projective module to the injective envelope of its top. Since A is connected, $f(\mathcal{U}) \neq \mathcal{U}$ and $f(\mathcal{N}) \neq \mathcal{N}$. So there are indices i, j such that e_iA is uniserial but e_jA is not, and soc $e_iA \cong S(j)$.

Let γ denote the unique arrow from i to j. The module e_jA is not uniserial, so we can find distinct arrows $\alpha : j \to {}'j$ and $\beta : j \to j'$. Note that both $\gamma\alpha$ and $\gamma\beta$ are 0. Considering the indecomposable injective modules $I({}'j)$ and $I(j')$ corresponding to the vertices ${}'j$ and j', we see that none of them is equal to $P(j)$, so α and β cannot be maximal nonzero paths from the right. It means that they are not maximal nonzero paths from the left either. So there exist arrows α', β' such that $\alpha'\alpha$ and $\beta'\beta$ are not 0. The arrows α', β' are distinct from γ (since $\gamma\alpha = \gamma\beta = 0$) and are also distinct from each other by the SB property of A. So there would be three distinct arrows ending at j, which is impossible in the quiver of a biserial algebra.

Finally, since A is a connected and self-injective algebra with only two-dimensional indecomposable projective modules, the quiver of A is indeed a directed cycle. $\square$

Remark 2.5. Note that in the second part of the proof, we have actually shown that if A does not have any uniserial projective module with dimension greater than 2, but possesses a uniserial projective module with dimension 2, then A must be an algebra described above.

We would like to show that a similar statement holds for standard Koszul self-injective algebras, namely, if the module e_iA with dimension at least 3 is uniserial, then A cannot be standard Koszul. That would mean – aside from the cases above – that it is enough to investigate algebras for which all the vertices in Γ have in- and out-degree 2.

Before we do that, we would remind the reader that the properties SB, self-injective and symmetric are "side-independent". That is, an algebra A is SB/self-injective/symmetric if and only if A° is SB/self-injective/symmetric. Due to the results of Green and Martínez-Villa [7], we know that A is quasi-Koszul if and only if A° is quasi-Koszul. We will use these facts later on.

Lemma 2.6. *If A is a non-simple standard Koszul algebra having a uniserial projective module, then every indecomposable projective A-module is two-dimensional.*

Proof. Suppose that e_iA is uniserial with $\dim_K e_iA \geq 3$ and soc $e_iA \cong S(t)$. Now, A° is also self-injective SB, and Ae_t is a uniserial A°-module. Since $\Delta(i)$ and $\overline{\Delta}^\circ(t)$ are in $\mathcal{C}_A^1$ and $\mathcal{C}_{A^\circ}^1$, respectively, both of them have to be either simple or projective. If $\Delta(i)$ is projective, then $\overline{\Delta}^\circ(t)$ is not, hence it is simple. Both cases contradict Lemma 2.3.

If A is a non-local, then Remark 2.5 gives that $A \cong K\Gamma/I$, where Γ is a directed cycle on n vertices and I is generated by all the paths of length 2. So every indecomposable projective module is two-dimensional. $\square$

In Proposition 2.7, we summarize the previous lemmas and observations.

Proposition 2.7. *If A is a standard Koszul self-injective SB algebra that has a uniserial projective module, then A is quasi-Koszul.*

Proof. The algebras K and $K[x]/\langle x^2\rangle$ are Koszul, while the algebras described in Proposition 2.4. are quadratic and monomial, hence Koszul, cf. [8]. $\square$

Now, we can move on, and may assume that Γ consists of vertices with in- and out-degree 2. We construct the minimal projective resolutions of simple modules over algebras of this type. Although our description uses Loewy diagrams – similarly to the diagrammatic method of [3] – we also give explicit calculations. These extend the result of Proposition 3.3 b) in [3].

As an aid for describing the projective resolutions of simple modules, we define a graph on $\mathbb{N}\times\mathbb{N}$ for each simple module $S(s)$ (Fig. 1).

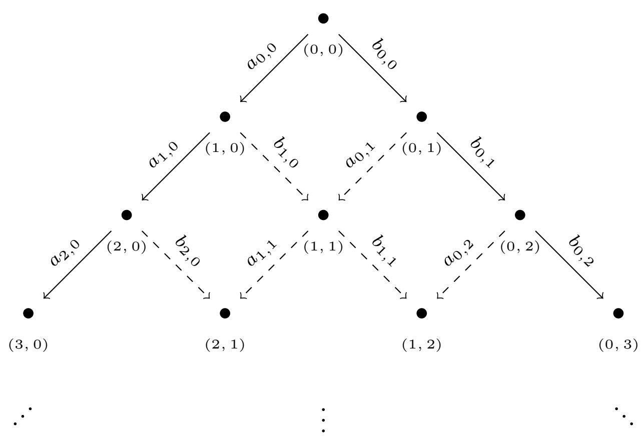

Figure 1: the graph corresponding to the resolution of $S(s)$

The origin, $(0,0)$ represents the vertex $s\in\Gamma$. For all pairs (i,j), there are exactly two arrows starting at (i,j): $a_{i,j}$ to $(i+1,j)$ and $b_{i,j}$ to $(i,j+1)$. The arrows represent scalar multiples of nonzero paths in the following way. First, $a_{0,0}$ and $b_{0,0}$ are just the two arrows starting at s. The leftmost and the rightmost arrows are defined recursively such that $a_{i+1,0}$ is the unique arrow of Γ for which $a_{i,0}a_{i+1,0}$ is a minimal zero path. The definition of $b_{0,i+1}$ is analogous. The other arrows are defined via the function δ. Let $a_{i,j}$ be $\delta(b_{i,j-1})$, for $j\geq 1$, and similarly, $b_{i,j}=\delta(a_{i-1,j})$ for $j\geq 1$. Let the common endpoint of the paths determined by $a_{i-1,j}$ and $b_{i,j-1}$ correspond to (i,j). We will denote the indecomposable projective module corresponding to the vertex assigned to (i,j) by $P(i,j)$.

Proposition 2.8 links this graph of s with the projective resolution of the simple module $S(s)$, namely, it will turn out that the hth term in the projective resolution of $S(s)$ is isomorphic to the direct sum of the indecomposable projective modules corresponding to the vertices contained in the hth level. Moreover, the syzygies can be tracked by stepping downwards on the levels of the graph.

Proposition 2.8. *Let s be a vertex in the quiver Γ of A, where Γ consists of vertices with in- and out-degree 2. The Loewy-diagram of the hth syzygy of $S(s)$ follows the row*

$$
\begin{tikzcd}[column sep=small]
\overset{(h,0)}{\bullet} \arrow[dr, dashed, "b_{h,0}"] & & \overset{(h-1,1)}{\bullet} \arrow[dl, dashed, "a_{h-1,1}"'] \arrow[dr, dashed, "b_{h-1,1}"] & & \overset{(h-2,2)}{\bullet} \arrow[dl, dashed, "a_{h-2,2}"'] & \cdots & \overset{(2,h-2)}{\bullet} \arrow[dr, dashed, "b_{2,h-2}"] & & \overset{(1,h-1)}{\bullet} \arrow[dl, dashed, "a_{1,h-1}"'] \arrow[dr, dashed, "b_{1,h-1}"] & & \overset{(0,h)}{\bullet} \arrow[dl, dashed, "a_{0,h}"'] \\
& \underset{(h,1)}{\bullet} & & \underset{(h-1,2)}{\bullet} & & & & \underset{(2,h-1)}{\bullet} & & \underset{(1,h)}{\bullet} &
\end{tikzcd}
$$

of Fig. 1. In fact, the hth term of the minimal projective resolution of the simple module $S(s)$ is

$$\mathcal{P}_h = \bigoplus_{i=0}^{h} P(i, h-i), \tag{1}$$

while the corresponding kernel is

$$\Omega_{h+1} = \tilde{a}_{h,0}A + \sum_{i=0}^{h-1} \left(b_{h-i,i} + \tilde{a}_{h-i-1,i+1}\right) A + b_{0,h}A, \tag{2}$$

where $\tilde{a}_{h-i,i} = (-1)^h a_{h-i,i}$.

Proof. We prove the statement by induction on h. For $h = 0$, it is obvious that $\mathcal{P}_0 = P(0,0)$ and $\Omega_1 = a_{0,0}A + b_{0,0}A$. Let us assume that our statement holds for some h.

Let $d_{h+1} : \bigoplus_{i=0}^{h+1} P(i, h+1-i) \to \Omega_{h+1}$ be described by the matrix

$$d_{h+1} = \begin{bmatrix} \tilde{a}_{h,0} & b_{h,0} & & & & & & & \\ & \tilde{a}_{h-1,1} & b_{h-1,1} & & & & & & \\ & & \ddots & \ddots & & & & & \\ & & & \tilde{a}_{h-i,i} & b_{h-i,i} & & & & \\ & & & & \ddots & \ddots & & & \\ & & & & & \tilde{a}_{1,h-1} & b_{1,h-1} & \\ & & & & & & \tilde{a}_{0,h} & b_{0,h} \end{bmatrix}$$

acting from the left. The map d_{h+1} is clearly surjective, and it is easy to check from Fig. 1 that

$$\tilde{a}_{h+1,0}A + \sum_{i=0}^{h} \left(b_{h+1-i,i} + \tilde{a}_{h-i,i+1}\right) A + b_{0,h+1}A$$

is in the kernel $\ker d_{h+1}$. On the other hand, for an arbitrary $x \in \ker d_{h+1}$, we have

$$\tilde{a}_{h-i,i}[x]_i + b_{h-i,i}[x]_{i+1} = 0$$

for all i, so

$$\tilde{a}_{h-i,i}[x]_i = -b_{h-i,i}[x]_{i+1} \in \operatorname{soc} P(h-i,i),$$

thus $[x]_i \in \operatorname{rad} P(h+1-i,i)$ for all i. This implies that d_{h+1} is the projective cover of Ω_{h+1}. Furthermore,

$$[x]_i = \alpha_{h-i+1,i} r_i + \beta_{h-i+1,i} s_i, \quad i = 0, \dots, h+1,$$

for some $s_i, r_i \in A$, where $\alpha_{h-i+1,i}$ and $\beta_{h-i+1,i}$ are the two distinct arrows (starting at the vertex $(h-i+1,i)$) in the ith summand of $\mathcal{P}_{h+1}$. Note that $\alpha_{h-i+1,i}$ and $\beta_{h-i+1,i}$ are the starting segments of $a_{h-i+1,i}$ and $b_{h-i+1,i}$, respectively, so both $\tilde{a}_{h-i,i}\alpha_{h-i+1,i}$ and $b_{h-i,i}\beta_{h-i,i+1}$ are zero according to the graph defined in Fig. 1. Let us compute $[d_{h+1}x]_i$.

$$[d_{h+1}x]_i = \tilde{a}_{h-i,i}\beta_{h-i+1,i} s_i + b_{h-i,i}\alpha_{h-i,i+1} r_{i+1} = 0. \tag{3}$$

The two terms in (3) have their images in distinct components of the factor $\operatorname{rad} \mathcal{P}_h / \operatorname{soc} \mathcal{P}_h$, so both terms must be in the socle of $\mathcal{P}_h$. Hence,

$$\beta_{h-i+1,i} s_i = b_{h-i+1,i} s'_i \quad \text{and} \quad \alpha_{h-i,i+1} r_{i+1} = a_{h-i,i+1} r'_{i+1}.$$

By the definitons of the monomials a, b and the function δ, we can write

$$\tilde{a}_{h-i,i} b_{h-i+1,i} s'_i = (-1)^h b_{h-i,i} a_{h-i,i+1} s'_i.$$

Therefore, after rewriting (3),

$$b_{h-i,i} a_{h-i,i+1}((-1)^h s'_i + r'_{i+1}) = 0,$$

that is, $(-1)^h s'_i + r'_{i+1} = j \in \operatorname{Ann}_r(b_{h-i,i} a_{h-i,i+1})$. Let us express j as a linear combination of paths u_i of A, and separate the terms:

$$j = \sum \lambda_i u_i = \underbrace{\sum_{a_{h-i,i+1} u_i = 0} \lambda_i u_i}_{j'} + \underbrace{\sum_{a_{h-i,i+1} u_i \neq 0} \lambda_i u_i}_{j''}.$$

Note that $b_{h-i+1,i} j'' = 0$ by the SB property, so we get

$$[x]_i = \alpha_{h-i+1,i} r_i + b_{h-i+1,i}(s'_i + (-1)^{h+1} j'')$$

and

$$[x]_{i+1} = a_{h-i,i+1}((-1)^{h+1} s'_i + j'') + \beta_{h-i+2,i} s_{i+1},$$

and on the right-hand side of the latter, the first term is $\tilde{a}_{h-i,i+1}(s'_i + (-1)^{h+1} j'')$. Applying this for every index i, we see that

$$\ker d_{h+1} \leq \tilde{a}_{h+1,0} A + \sum_{i=0}^{h} (b_{h-i+1,i} + \tilde{a}_{h-i,i+1}) A + b_{0,h+1} A$$

because at both ends, $a_{h+1,0}$ and $b_{0,h+1}$ are just the arrows $\alpha_{h+1,0}$ and $\beta_{0,h+1}$. □

Using the graph defined above, we can now give a characterization of quasi-Koszul self-injective SB algebras.

Proposition 2.9. *With notations as above, A is quasi-Koszul if and only if for all vertices s in Γ, one of the elements $a_{0,1}$ and $b_{1,0}$ is a scalar multiple of an arrow.*

Proof. First, it is easy to see that if for a vertex $s \in \Gamma$, both of $a_{0,1}$ and $b_{1,0}$ are in $\operatorname{rad}^2 A$, then $\Omega_2(s) \overset{t}{\nleq} \operatorname{rad} P_1(s)$, hence the simple module $S(s) \notin \mathcal{C}_A^2$.

Conversely, fix an arbitrary vertex s. It is enough to show that

$$a_{0,1} \text{ or } b_{1,0} \text{ is an arrow} \Rightarrow (\forall h : \operatorname{rad}^2 \mathcal{P}_h \cap \Omega_{h+1} \leq \operatorname{rad} \Omega_{h+1}).$$

Note that on the right-hand side of (2), each generating element (except $a_{h,0}$ and $b_{0,h}$) is a linear combination of two paths. One term of the combination is a scalar multiple of an arrow, otherwise these two paths would be the end segments of two distinct nonzero paths with length at least 3, ending at the same vertex. Then we could find an appropriate vertex s' for wich the condition of the statement fails.

Let h and $x \in \Omega_{h+1}$ be arbitrary. By using the form (2) of Ω_{h+1} and rearranging, we can express x as

$$x = \sum_{i=0}^{h} \tilde{a}_{h-i,i} r_i + b_{h-i,i} r_{i+1} \tag{4}$$

with appropriate elements $r_i \in A$. In (4), the members are sorted so that for each i, the term $\tilde{a}_{h-i,i} r_i + b_{h-i,i} r_{i+1}$ belongs to the same direct component of $\mathcal{P}_h$, therefore, $x \in \operatorname{rad}^2 \mathcal{P}_h$ means that all these terms are in the radical-square of the component they are contained in. Focusing on one particular term, we observe that its image in $\operatorname{rad} \mathcal{P}_h / \operatorname{soc} \mathcal{P}_h$ is a combination of two elements from distinct components, so each member of this sum has to be in the radical-square, hence $r_i \in \operatorname{rad} A$ for all $i \geq 1$. □

Corollary 2.10. *A is quasi-Koszul if and only if all of its simple modules are in $\mathcal{C}_A^2$.*

Proof. One direction is obvious. For the converse, we observe that if both elements $a_{0,1}$ and $b_{1,0}$ are in $\operatorname{rad}^2 A$, then $S(s) \notin \mathcal{C}_A^2$. □

In the remaining part of the section, we will prove that a standard Koszul self-injective SB algebra satisfies the conditions of Proposition 2.9 for being quasi-Koszul. Without loss of generality we may assume that there are no uniserial projective modules.

Definition 2.1. Let $u = s_1 \to s_2 \to \ldots \to s_m$ be a path in Γ. We say that u contains a *valley* if there exist indices $j < k < \ell$ for which $s_j \geq s_k$ and $s_k < s_\ell$.

Lemma 2.11. *If $u = s_1 \to s_2 \to \ldots \to s_m$ contains a valley, then it also contains a short valley, that is, there exists an index i such that $s_{i-1} \geq s_i$ and $s_i < s_{i+1}$.*

Proof. Let $u' : s_j \rightsquigarrow s_k \rightsquigarrow s_m$ be the subpath of u with $j < k < m$ such that $s_j \geq s_k$ and $s_k < s_m$. Let t denote the largest index between j and m for which s_t is minimal. Then $s_{t-1} \to s_t \to s_{t+1}$ is a short valley. □

Lemma 2.12. *Let u be a nonzero path such that $|u|$ (the length of u) is at least 3. If u contains a valley, then A is not standard Koszul.*

Proof. Suppose that u starts at the vertex i. We may assume that e_iA is not uniserial (see Lemma 2.6). By the previous lemma, there is a subpath $u' : j \to k \to m$ of u, which is a short valley. If $j > m$, then $u'A$ is a top submodule of $\Omega(\Delta(j))$ but it is not a top submodule of rad e_jA, so $\Delta(j) \notin \mathcal{C}^1_A$. On the other hand, if $j \leq m$, then a similar argument shows that $\overline{\Delta}^{\circ}(m) \notin \mathcal{C}^1_{A^{\circ}}$. $\square$

Proposition 2.13. *Let A be standard Koszul. For all i, at least one of the maximal nonzero paths in e_iA has length 2.*

Proof. Assume - on the contrary - that e_iA contains two distinct paths u, v with length at least 3. Let t denote the common endpoint of u and v:

$$u : i \to {}'i \to {}''i \to \ldots \to {}^{(b)}i \to t$$
$$v : i \to i' \to i'' \to \ldots \to i^{(r)} \to t,$$

where $r, b \geq 3$.

Recall from the proof of Proposition 2.9 that if for some s, both $a_{0,1}$ and $b_{1,0}$ are in $\text{rad}^2 A$, then the simple module $S(s)$ fails to be in $\mathcal{C}^2_A$. Therefore, $S(i) \notin \mathcal{C}^2_A$ and $S^{\circ}(t) \notin \mathcal{C}^2_{A^{\circ}}$, so $\Delta(i) \neq S(i)$ and $\overline{\Delta}^{\circ}(t) \neq S^{\circ}(t)$. These also imply that none of $\Delta(i)$ and $\overline{\Delta}^{\circ}(t)$ can be projective. For example, if $\Delta(i)$ is projective, then ${}^{(b)}i, i^{(r)} \leq i$, and since $\overline{\Delta}^{\circ}(t) \neq S^{\circ}(t)$, one of $i^{(r)}$ and ${}^{(b)}i$ – let us say ${}^{(b)}i$ – must be less than t. But then u contains a valley, contradicting Lemma 2.12. Similarly, if $\overline{\Delta}^{\circ}(t) = Ae_t$, then u or v contains a valley.

So neither $\Delta(i)$ nor $\overline{\Delta}^{\circ}(t)$ is simple or projective, hence exactly one of i' and ${}'i$ is greater than i, and exactly one of ${}^{(b)}i$ and $i^{(r)}$ is less than t.

Since none of the paths u and v contains a valley, it can be assumed that the indices are increasing along u and decreasing along v. That would mean that both $i < t$ and $t < i$, a contradiction. $\square$

Theorem 2.14. *If A is a self-injective standard Koszul SB algebra, then A is quasi-Koszul.*

Proof. The theorem is an easy consequence of Propositions 2.7, 2.9 and 2.13. $\square$

3 Standard Koszul symmetric special biserial algebras

We now turn our attention to SSB algebras. In this section, let $A = K\Gamma/I$ be a standard Koszul SSB algebra. Recall that the existence of the symmetric form φ implies that soc A_A is generated by cycles. Moreover, if the path $u = \alpha_1, \alpha_2 \ldots, \alpha_k$ is in soc A_A, then the paths $\alpha_i, \alpha_{i+1}, \ldots, \alpha_{i-1}$ are in the socle of A_A for all i.

Observe that if e_iA is an indecomposable projective A-module, and all composition factors of e_iA are isomorphic to $S(i)$, then the fact that A is connected implies that $A_A = e_iA$, i.e. A is local.

First, we handle non-local algebras; local algebras will be discussed later. So from now on, A possesses at least two non-isomorphic simple modules, and for every i, there exists $j \neq i$ such that $S(j)$ is a composition factor of e_iA. Thus $\dim_{\mathrm{K}} e_iA \geq 3$ for every i. This condition, along with the former results of the paper, implies the next statement.

Proposition 3.1. *If A is non-local, then no e_iA is uniserial.*

We can use combinatorial arguments again to characterize the quivers Γ for which $K\Gamma/I$ can be standard Koszul. To obtain the generating relations of I for such quivers, we have to use only that A is symmetric (i.e. the existence of some symmetric form φ).

Proposition 3.2. *If A is standard Koszul, and u is a maximal nonzero path in A with $|u| \geq 3$, then u is a power of a loop.*

Proof. Let $u : u_1 \to u_2 \to \ldots \to u_m \to u_1$ be a maximal nonzero path with $m \geq 3$. Suppose that u passes through at least two distinct vertices. Let u_k be a maximal vertex in u. Now, $u' : u_k \to u_{k+1} \to \ldots \to u_{k-1} \to u_k$ is also a maximal nonzero path. Since u_k is maximal, the path u' contains a valley, contradicting Lemma 2.12. $\square$

Corollary 3.3. *If A is non-local, then all the vertices in Γ are contained in exactly two maximal nonzero cycles. If one of these cycles has length greater than 2, then it is a power of a loop.*

Lemma 3.4. *If $\alpha : a \to b$ is an arrow in Γ with $a \neq b$, then there must also exist a unique arrow $\beta : b \to a$. Moreover, both $\alpha\beta$ and $\beta\alpha$ are maximal nonzero paths.*

Proof. The existence of such β is just a reformulation of Proposition 3.2 and Corollary 3.3. For the uniqueness, suppose that there are two arrows $\beta_{1,2} : b \to a$. Using the connectedness and the SSB property of A, along with Corollary 3.3, we see that Γ consists of two vertices and the four arrows $\alpha_{1,2} : a \to b$ and $\beta_{1,2} : b \to a$. It follows from Corollary 3.3 that $\overline{\Delta}^{\circ}(2) = Ae_2/\operatorname{soc} Ae_2$ but this is not in $\mathcal{C}^1_{A^\circ}$. $\square$

Lemma 3.5. *Γ contains at most 2 vertices that are endpoints of loops.*

Proof. Let us assume that there are (at least) three vertices in Γ that are endpoints of loops. Let them be s_1, s_2, s_3. Since A is connected, there are directed paths from s_1 to s_2 and from s_1 to s_3. Let u_1 and u_2 be two such paths with minimal length. Let t be the vertex where u_1 and u_2 differ for the first time. Using Lemma 3.4, it is easy to check that both the in- and out-degree of t is at least 3, and this contradicts the SB property of A. $\square$

Lemma 3.6. *If A is non-local, then Γ contains exactly 2 vertices that are endpoints of loops.*

Proof. First, suppose that there are no loops in Γ. We take an arbitrary vertex s_0. Since A is connected, s_0 has a neighbour, i.e. there exists an arrow $\alpha_1 : s_0 \to s_1$ (and by Lemma 3.4 an arrow $\beta_1 : s_1 \to s_0$, too). The vertex s_1 also has out-degree 2, so there must be an arrow $\alpha_2 : s_1 \to s_2 \neq s_0$, and so on. At some point, s_n coincides with s_0. This means that the arrows $\alpha_0, \alpha_1, \ldots, \alpha_n$ and $\beta_0, \beta_1, \ldots, \beta_n$ in Γ form two disjoint, parallel and oppositely directed cycles on n vertices, and the maximal nonzero paths of A are of the form $\alpha_i \beta_i$ and $\beta_i \alpha_i$. One can check that $\overline{\Delta}^{\circ}(n) = P^{\circ}(n)/\operatorname{soc} P^{\circ}(n) \notin \mathcal{C}^1_{A^{\circ}}$.

We can repeat the first part of the previous argument in the situation where Γ contains exactly 1 loop. Now, starting with the vertex s_0 (which belongs to the loop) and running over the vertices $s_1, s_2, \ldots$ will lead us to some vertex s_n that has to coincide with a former vertex. But that vertex would have in- and out-degree (at least) 3. □

Corollary 3.7. *If Γ has at least 2 vertices (i.e. A is not local), then Γ has the shape shown in Fig. 2.*

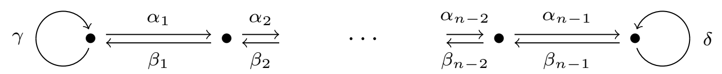

Figure 2: the quiver of a non-local standard Koszul SSB algebra

Proposition 3.8. *Let $A = K\Gamma/I$ be a non-local standard Koszul SSB algebra. Then Γ has the shape shown in Fig. 2, and I is generated by the relations*

$$I = \big\langle \alpha_i\alpha_{i+1},\ \beta_{i+1}\beta_i,\ \gamma\alpha_1,\ \alpha_{n-1}\delta,\ \beta_1\gamma,\ \delta\beta_{n-1},\ \alpha_{i+1}\beta_{i+1} - \beta_i\alpha_i,$$
$$\gamma^k - \alpha_1\beta_1,\ \lambda\delta^m - \beta_{n-1}\alpha_{n-1} \mid i = 1, \ldots, n-2 \big\rangle,\ \lambda \in K \setminus \{0\},\ k, m \geq 2. \quad (5)$$

The vertices are indexed so that the indices are increasing from the vertex with the smallest index towards either end according to Fig 2.

Proof. We may assume that I is generated by paths and differences of paths except the term $\lambda\delta^m - \beta_{n-1}\alpha_{n-1}$. Otherwise, we can exchange the arrows for their scalar multiples repeatedly (let us say moving from left according to the quiver in Fig. 2). At the last vertex, we might not be able to do this if K is not algebraically closed.

If the ordering differs from the one we stated, then there is a vertex k having neighbours only with lower indices. In this situation $\overline{\Delta}^{\circ}(k) \notin \mathcal{C}^1_{A^{\circ}}$. □

Let us investigate now the local algebras. There are two cases depending on the degree of the single vertex $s \in \Gamma$. If there is only one arrow in Γ, then A is monomial and commutative. Therefore A is standard Koszul if and only if $\overline{\Delta}(1) = S(1) \in \mathcal{C}_A$. Hence $A \cong K$ or $K[x]/\langle x^2 \rangle$.

For the other case, suppose that $s \in \Gamma$ has degree 2, and let the two arrows be x and y. If $xy \neq 0$, then $x^2 = 0$ and $y^2 = 0$ by the SB property. We show that $xyx = 0$. Assume – on the contrary – that $xyx \neq 0$. If $xyxy = 0$, then $\varphi(xyxy) = \varphi(yxyx) = 0$, and $AxyxA$ would be a proper ideal in $\ker\varphi$ because we have $\varphi(xyx) = \varphi(x^2y) = 0$. Otherwise, if $xyxy \neq 0$, then both $\dim_{\mathrm{K}} yA$ and $\dim_{\mathrm{K}} xA \geq 3$, hence A is not standard Koszul (cf. Lemma 2.13). Therefore if $xy \neq 0$, then $xyx = 0$, and similarly, $yxy = 0$. Since A is symmetric, $\varphi(xy - yx) = 0$, and $x^2 = y^2 = xyx = yxy = 0$ implies that the ideal generated by $xy - yx$ is in $\ker\varphi$, so $xy = yx$. Consequently, $A \cong K[x, y]/\langle x^2, y^2\rangle$.

Suppose that $xy = 0$. Then $yx = 0$ (otherwise $AyxA$ is a proper ideal in $\ker\varphi$), but since the socle is simple, $x^2, y^2 \neq 0$. Besides, if k, m are the smallest integers such that x^k and y^m are in the socle, then $I = \langle xy, yx, x^k - \lambda y^m\rangle$, where $\lambda \in K \setminus \{0\}$. Since A is standard Koszul, we may assume, for example, that $k = 2$.

Theorem 3.9. *$A = K\Gamma/I$ is a standard Koszul SSB algebra if and only if either*

(a) A is isomorphic to one of the K-algebras: K, $K[x]/\langle x^2\rangle$, $K[x, y]/\ \langle x^2, y^2\rangle$, $K\langle x, y\rangle/\langle xy, yx, x^2 - \lambda y^m\rangle$, or

(b) Γ has the shape shown in Fig. 2 and I is generated by the relations of (5).

Proof. In the local case, we have shown that the conditions of (a) are necessary. Note that if A is local, then it is standard Koszul if and only if it is quasi-Koszul. One may apply our former observations or Proposition 2.9 to the algebras described in (a), and see that they are standard Koszul. In the non-local case, we have seen that the conditions of (b) are necessary.

Suppose now that the condition (b) holds for A. We show that both $\Delta(i)$ and $\overline{\Delta}^{\circ}(i)$ are quasi-Koszul for all i. We may assume that none of them is projective or simple. (Projective modules are in $\mathcal{C}_A$, and if A satisfies the conditions of (b), then it is quasi-Koszul by Proposition 2.9, so all its simple left and right modules are quasi-Koszul.) If $\Delta(i)$ (or $\overline{\Delta}^{\circ}(i)$) is neither simple nor projective, then we can see from the induction step in the proof of Lemma 2.2 that each of the syzygies $\Omega_h(\Delta(i))$ (or $\Omega_h(\overline{\Delta}^{\circ}(i))$) is generated by a respective arrow for all h. According to Lemma 2.1, these submodules are top submodules of the radical of the projective module that they are contained in.

The only thing to check is whether these algebras are symmetric, since they are obviously SB. For the algebras in (a), let φ be 1 on an arbitrary basis element of $\operatorname{soc} A_A$ and 0 on all the other subspaces generated by paths. For the algebras in (b), define φ to be 1 on all the maximal nonzero paths of A except that $\varphi(\delta^m) = 1/\lambda$. Let φ vanish on all the other paths. It is easy to check that these functions can be extended to symmetric forms for the given algebras. $\square$

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