Linear series on surfaces and Zariski decomposition

This is an extended version of a talk given at the Algebra-Geometry seminar at the university of Freiburg in May 2011.

1 Zariski decomposition for effective divisors

Zariski decomposition is a way to dismantle orthogonally (pseudo-)effective divisors into positive and negative parts, both of which enjoy the useful geometric properties. For example, the birational map ? to D is largely determined by it's positive part, while the curves contracted by the morphisms all lie in the negative part of D.

Theorem 1. *Let D be an effective $\mathbb{Q}$-divisor on a smooth projective surface X. Then there exist uniquely determined effective $\mathbb{Q}$-divisors P and N with*

$$D = P + N$$

such that

i) P is nef

ii) N is zero or has negative definite intersection matrix

iii) $P \cdot C = 0$ for every irreducible component C of N

Remark 1. *We quickly recall a couple of definitions Let $Div_{\mathbb{Q}}(X) := Div(X) \otimes \mathbb{Q}$. On smooth projective surfaces all $\mathbb{Q}$-Weil divisors are also $\mathbb{Q}$-Cartier, hence we can write $\mathbb{Q}$-Cartier every divisor D as a finite sum $\sum x_i C_i$, where the C_i are distinct integral curves and $x_i \in \mathbb{Q}$. A divisor D is called effective (or sometimes positive) if $x_i \geq 0 \ \forall i$. If $D \cdot C \geq 0$ for all integral curves C then D we be called nef. For a divisor $D = \sum x_i C_i$ we define the intersection matrix to by the symmetric matrix $I(D) = I(C_1, \ldots, C_q) = (C_i \cdot C_j)_{i,j}$. By definition we have $D \cdot D = x^t \cdot I(D) \cdot x$, where $x = (x_1, \ldots, x_n)^t$. We will call P a sub divisor of D, if $D - P$ is effective or zero. We will then write $P \preceq D$.*

Example 1. a) *The group of linear equivalence classes of Weil Divisors on $\mathbb{P}^2$ is isomorphic to $Pic(\mathbb{P}^2)) \cong \mathbb{Z}$. The line h corresponds to the divisor generating $Pic(\mathbb{P}^2)$, since all lines in $\mathbb{P}^2$ are linear equivalent to each other and different lines intersect exactly in one point we have $h \cdot h = 1$. Thus the nef divisors are exactly the positive ones.*

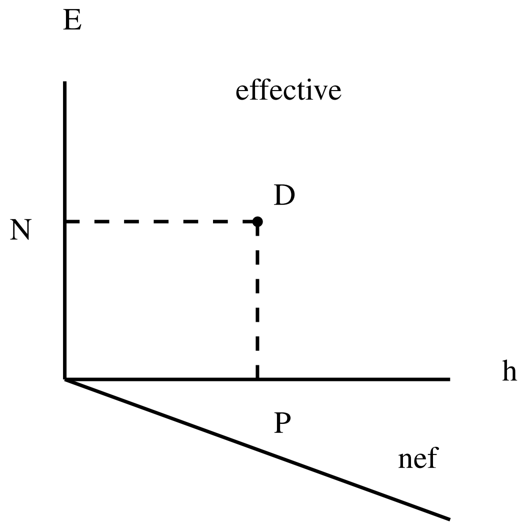
b) *In the case of $\mathbb{P}^2$ blown up in one point p we have the following possibilities of curves:*

i) E the blow up of p

ii) nh is the pullback of a curve in $X - p$

iii) L is the proper transform of a curve in X with $p \in \pi_* L$, in this case we can write it in the form of $nh - E$

*To obtain the nef cone (**Picture**) we only need to determine the $ah+bE$ which intersect those curves above positively or zero, these are the points $a+b \geq 0$ with $a \geq 0$ and $-b \geq 0$. Now we see, that if $D = ah+bE$ is effective, then ah is the positive part of the Zariski decomposition and bE the negative part.*



Example 2. *If D is an effective integral divisor then the positive or negative part of it's Zariski decomposition does not have to be integral anymore*

Before we are able to proof theorem 1 we need a technical lemma.

Lemma 1. *If N is a $\mathbb{Q}/\mathbb{R}$-divisor on a projective surface which intersection matrix is not negative definite, then there exists an effective nef divisor $E \neq 0$ which components are among those of N.*

Granting lemma 1 for a moment we will show how to prove theorem 1 following [BCK] closely.

Proof of Theorem 1. The idea of the proof is the following: First we will show that a maximal effective nef subdivisor of D exists and fulfills properties of P i) to iii). Then we will show that conditions i) to iii) force P to be maximal nef and conclude uniqueness.
Existence: Let be $D = \sum_{i=1}^n a_i C_i$ with $a_i > 0$. If P is an effective subdivisor of D then $P = \sum_{i=1}^r x_i C_i$ with $0 \leq x_i \leq a_i$. Now P is nef if and only if

$$\sum_{i=1}^{r} x_i C_i \cdot C_j \geq 0 \;\forall\; 1 \leq j \leq r \tag{1}$$

since by definition the intersection with an integral curve C not among the components of D is not negative, c.f. [Har] p 360.
Now let us allow $\mathbb{R}$-divisors. The set of all possible solutions of the inequalities (1) can be identified with $K := \bigcap_{j=1}^n \{(x_1, \ldots, x_n) \in [0, a_1] \times [0, a_n] | \sum_{i=1}^r x_i C_i \cdot C_j \geq 0\}$ which as an

intersection of convex sets is convex. It also a closed subset of the real cubic given above and so it is compact. Therefore there exists a maximal solution corresponding to the relation $\preceq$. We fix the real divisor $P = \sum_{i=1}^{r} x_i C_i$ corresponding to this solution and set $N = D - P$. At the end of the proof we will show that if we start with a $\mathbb{Q}$-divisor D, then P and N will also have to be $\mathbb{Q}$-divisors. By definition P is an effective, nef subdivisor of D, so N is also effective. This shows i).
To see ii) we assume that if $D \neq P$, that $I(N)$ is not negative definite, by lemma 1 there exists an E such that for a small rational $\varepsilon > 0$ $P' := P + \varepsilon E$ is still effective and nef. But since $P' \succeq P$ this contradicts the maximality condition on P.
iii) If C is a component of N with $P \cdot C > 0$, then there exists a rational $\varepsilon > 0$ such that $P + \varepsilon C \geq 0$ is still nef, but $P + \varepsilon C \succeq P$ this also contradicts the maximality condition on P.
Uniqueness: Let $D = P + N$ be a decomposition fulfilling properties i) to iii) of theorem 1, we will show that P is indeed a unique maximal subdivisor of D. If $P' \succeq P$ is another nef divisor, then $P' = P + \sum_{j=1}^{l} y_i C_j$, where $y_i \geq 0$ and C_j are the components of N. Since P' is nef we have

$$0 \leq P'C_i = \underbrace{P \cdot C_i}_{=0} + (\sum_{j=1}^{l} y_j C_j) \cdot C_i$$

On the other hand the negative definiteness of the intersection matrix of N forces

$$0 \leq (\sum_{j=1}^{l} y_j C_j)^2 = \sum_{k=1} y_k \underbrace{(\sum_{j=1}^{l} y_j C_j C_k)}_{\geq 0} \leq 0$$

therefore $y_k = 0$ and so $P = P'$.
Let $P = \sum_{i=1}^{n} x_i C_i$ and $P' = \sum_{i=1}^{n} y_i C_i$ maximal solutions of (1), then $P'' = \sum_{i=1}^{n} \max(x_i, y_i) C_i$ is also nef. To see this, we have to check that that $P'' \cdot C_j = 0$ for all j, this is equivalent to

$$\sum_{i=1, i \neq j}^{n} \max(x_i, y_i) C_i C_j \geq -\max(x_j, y_j) C_j \cdot C_j$$

now

$$-y_j C_j \cdot C_j \leq \sum_{i=1, i \neq j}^{n} y_i C_i C_j \leq \sum_{i=1, i \neq j}^{n} \max(x_i, y_i) C_i C_j \geq \sum_{i=1, i \neq j}^{n} x_i C_i C_j \geq -x_j C_i C_j$$

since $C_j \cdot C_i \geq 0$ for $i \neq j$. Therefore $P \preceq P'' \succeq P'$ and we conclude $P = P'' = P'$, this was to show!
It remains to show that P (and then automatically also N) is a $\mathbb{Q}$-divisor, provided D is. Let $D = \sum_{i=1}^{n} a_i C_i$ and $N = \sum_{j=1}^{r} b_i C_i$, with $r \leq n$. Let be $M = (C_i \cdot C_j)_{1 \leq i \leq r, 1 \leq j \leq n}$, then $M = (I(N)\ A)$[1] for some integral matrix A. If $P = \sum x_i C_i$, then ii) means $P \cdot C_i = 0$ which is equivalent to $M \cdot x = 0$. On the other hand we have $a_i = x_i$ for all $r + 1 \leq i \leq n$, this

[1]If $X = (x_{ij})$ is a $n \times k$ matrix and $Y = (y_{ij})$ is a $n \times l$ matrix, then $Z = (X\ Y) = (z_{ij})$ is defined to be the $n \times k + l$ matrix with $z_{ij} = x_{ij}$ if $j \leq k$ and $z_{ij} = y_{i(j-k)}$ for $j > k$.

means x is the solution of the system

$$\begin{pmatrix} I(N) & A \\ 0 & Id_{n-r} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_r \\ x_{r+1} \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

Now since $I(N)$ is negative definite the matrix on the left hand side has maximal rank, thus this solution is unique. The elements of the matrix are all integral. If D is a $\mathbb{Q}$-divisor, then all elements on the right hand side are rational, thus all the x_i are also rational as well. $\square$

Proof of Lemma 1. Let $N = \sum_{i=1}^n n_i C_i$ where C_i are distinct integral curves on X. $I(N)$ **is not negative semidefinite:**
In this case there exists a vector b with $b^t I(N) b > 0$, this vector corresponds to a divisor $B = \sum_{i=1} b_i C_i$ whose components are among those of N with $B^2 > 0$. If B is not effective, we can write $B = B_1 - B_2$ as the difference of two effective divisors having no common component. Since

$$0 < (B_1 - B_2)^2 = B_1^2 - 2B_1 \cdot B_2 + B_2^2$$

and $B_1 \cdot B_2 \geq 0$ we conclude that at least one $B_i \cdot B_i > 0$. By replacing B with a B_i we can assume that B is effective. Now since B is effective and $B^2 > 0$ the Riemann-Roch theorem gives

$$\begin{aligned} \frac{1}{2} mB(mB + K_X) + \chi(\mathcal{O}_X) &= h^0(X, \mathcal{O}_X(mB)) - h^1(X, \mathcal{O}_X(mB)) + h^2(X, \mathcal{O}_X(mB)) && (2) \\ &= h^0(X, \mathcal{O}_X(mB)) - h^1(X, \mathcal{O}_X(mB)) + h^0(X, \mathcal{O}_X(K_X - mD)) && (3) \\ &\leq h^0(X, \mathcal{O}_X(mB)) \text{ for } m \gg 0 && (4) \end{aligned}$$

where we used Serre Duality in (2) and the fact that $K_X - mD$ cannot be effective for $m \gg 0$ in (3). Now the left hand side in (1) can be written in the form of

$$\frac{1}{2} B^2 m^2 + b \cdot m + c$$

where $b := \frac{1}{2} mBK_X$ and c are rational (or real it does not matter) numbers, thus B is a big. So we can write $lB = E_l + F_l$ where E_l is the moving part and F_l the fixed component part of lB. On the other hand $|B_l| = |E_l| + F_l$, bigness of B implies that $|E_l| \neq \varnothing$ for l large enough, thus by [Bad] p. 217 $E := E_l$ is nef and we are done.
$I(N)$ **is negative semidefinite:**
We will do the proof by induction on the number of components. Say $N = \sum_{i=1}^n a_i C_i$. If $n = 1$ then $N^2 = 0 = C_1^2$, then N is nef. For $n > 1$ we will find a non trivial divisor $R := \sum_{i=1}^n r_i C_i$ corresponding to $r^t \cdot I(N) r = 0$, thus $R \cdot R = 0$. If R or $-R$ is effective, we are done, if not we will write $R = R_1 - R_2$ as the difference of effective divisors having no common component. Then $0 = R^2 = R_1^2 - 2R_1 R_2 + R_2^2$, since $R_i^2 \leq 0$ we get $R_i^2 = 0$. Thus at least one of the R_i's is a non trivial divisor having negative semi-definite intersection matrix and fewer components than R, by induction hypothesis the desired divisors exists. $\square$

2 Fujita's generalization to Pseudo-effective divisors

Now we will give an analogues statement to Theorem 1 for pseudo-effective divisors.
Recall that by the Nakai-Moishezon criterion a divisor H on a smooth projective surface is ample if and only if $H \cdot H > 0$ and $H \cdot C > 0$ for every integral curve C.

Definition 1. *As before X denotes a smooth projective surface. A divisor D is called pseudo-effective if $D \cdot H \geq 0$ for every ample divisor H.*

One can show for example, that a divisor D is pseudo-effective if $D \cdot N \geq 0$ for all nef divisors N.

Remark 2. *Using the Nakai-Moishezon criterion we see that the closure of the effective cone (i.e. all divisors numerically equivalent to an effective divisor) in $NS_{\mathbb{R}}(X)$ is equal to the set of pseudo-effective divisors.*

Example 3. *We want to give an example of an pseudo-effective divisor which is not effective. Let C be an elliptic curve of degree 0 and ρ be a divisor which is no torsion element in $Pic(C)$, i.e. $\rho = p_1 - p_2$, $p_i \in C$. Then $\rho \times \mathbb{P}^1$ is is not an effective divisor of $X := C \times \mathbb{P}$, but corresponds to 0 in $NS_{\mathbb{R}}(X)$, therefore it is pseudo-effective.*

Theorem 2. *Let D be a pseudo-effective $\mathbb{Q}$-divisor on a smooth projective surface X. Then there exist uniquely determined $\mathbb{Q}$-divisors P and N with*

$$D = P + N$$

such that

i) P is nef

ii) N is zero or is effective and has negative definite intersection matrix

iii) $P \cdot C = 0$ for every irreducible component C of N

Remark 3. *By definition a $\mathbb{R}$-divisor $D_{\mathbb{R}}$ is a $\mathbb{Q}$-divisor $D_{\mathbb{Q}}$ multiplied by some real number a. The Zariski decomposition of $D_{\mathbb{R}}$ is then defined by $D_{\mathbb{R}} = a \cdot P_{D_{\mathbb{Q}}} + aN_{D_{\mathbb{Q}}}$, i.e. the theorem above also holds for pseudo-effective $\mathbb{R}$-divisors.*

Before we are able to proof this theorem we need to list some facts.

Lemma 2. *Let X be a smooth, projective surface and $C_1, \ldots, C_q$ integral curves on X such that the intersection matrix $(C_i \cdot C_j)_{i,j}$ is negative definite. Let $D = \sum_{i=1}^{q} a_i C_i$ be a $\mathbb{Q}$-divisor on X*

i) If $D \cdot C_j \leq 0$ for all j , then $D \geq 0$

ii) If there exists a pseudo-effective divisor D' with $(D' - D) \cdot C_j \leq 0$ for all j, then $D' - D$ is pseudo-effective.

Proof. [Bad] Lemma 14.9 and 14.10. □

Lemma 3. *Let $C_1, \ldots, C_q$ be curves on a smooth projective variety X, and let D be a pseudo-effective $\mathbb{Q}$-divisor such that $D \cdot C_i \leq 0$, $\forall 1 \leq i \leq q$. Assume there exists an r such that $1 \leq r < q$ and $D \cdot C_j < 0$, $\forall r+1 \leq j \leq q$. If the intersection matrix $I(C_1, \ldots, C_r)$ is negative definite, then the intersection matrix $I(C_1, \ldots, C_q)$ is also negative definite.*

Proof. [Bad] lemma 14.12. □

Corollary 1. *If D is a pseudo-effective divisor, then there exist only finitely many integral curves C_i such that $D \cdot C_i < 0$ and the intersection matrix of $C_1, \dots, C_q$ is negative definite.*

Proof of theorem 2. We will only sketch the proof of the existence, which contains an algorithm to obtain the Zariski decomposition. Let $C_1 \cdots, C_{q_1}$ be all curves with $D \cdot C_i < 0$. By corollary 1 there are only finitely many of these and the intersection matrix of $\sum_i C_i$ is negative definite. Then there exists exactly one divisor $N_1 = \sum_{i=1}^{q_1} b_i C_i$ such that $N_1 \cdot C_i = D \cdot C_i$, in fact the vector $b := (b_1, \dots, b_{q_1})$ is the (unique) solution of the system of linear equations $(C_i \cdot C_j)_{i,j} b = (D \cdot C_1, \dots, D \cdot C_{q_1})$ since negative definiteness implies that the intersection matrix has full rank! By lemma 2 i) $N_1 \geq 0$. Set $D_1 = D - N_1$. If D_1 is not nef then it still has to be pseudo-effective by lemma 2 ii). Let $C_{q_1+1}, \dots C_{q_2}$ be all curves such that $D_1 \cdot C_j < 0$ for all $q_1 < j \leq q_2$, then the intersection matrix of $\sum_{i=1}^{q_2} C_i$ is negative definite by lemma 3, and we construct N_2 as above. We are done if we can show, that the procedure has to terminate after finitely many steps. To see this note that because of the negative definiteness of the intersection matrix the curves C_i define linear independent elements in the Néron-Severi group $NS(X)_{\mathbb{Q}}$, which has by the theorem of the base only finite rank, so we are done. (See for example the end of the proof of corollary 1) □

Proof of Corollary 1. Let $C_1, \dots, C_q$ be integral curves in X such that $D \cdot C_i < 0, \forall\, 1 \leq i \leq q$, then by lemma 3 the intersection matrix $I(C_1 \dots, C_q)$ is negative definite. Now we will show that C_i are linearly independent elements of the Neron Severi Group $NS(X)_{\mathbb{Q}}$. Assume there exist $a_i \in \mathbb{Q}$ such that $(\sum_{i=1}^{q} a_i C_i) \cdot D' = 0$ for all D', therefore for $a = (a_1, \dots, a_q)$ we have $0 = (\sum_{i=1}^{q} a_i C_i)^2 = a^t \cdot I(C_1, \dots, C_q) \cdot a \leq 0$. Since $I(C_1, \dots, C_q)$ is negative definite, this can only happen when $a = 0$, so $C_1, \dots, C_q$ are linearly independent in $NS(X)_{\mathbb{Q}}$. By the theorem of the base (see [PAG], proposition 1.1.16) $NS(X)_{\mathbb{Q}}$ is a free abelian group of finite rank, this implies $q \leq \operatorname{rk} NS(X)_{\mathbb{Q}}$ and has therefore to be finite. □

For a real number $x \in \mathbb{R}$ we define $\lceil x \rceil := \min\{n \in \mathbb{Z} | n \geq x\}$. If $D := \sum_i a_i C_i$ is a $\mathbb{Q}$ or $\mathbb{R}$-divisor, then $\lceil D \rceil = \sum_i \lceil a_i \rceil C_i$.

Proposition 1. *Let D be an integral effective divisor, if $D = P + N$ is it's Zariski decomposition, then the canonical map*

$$H^0(X, \mathcal{O}_X(mD - \lceil mN \rceil)) \longrightarrow H^0(X, \mathcal{O}_X(D))$$

is bijective.

Proof. See [PAG], proposition 2.3.21 □

Corollary 2. *Let D be a pseudo-effective $\mathbb{Q}$-divisor, then*

$$vol(D) = vol(P_D) = P_D \cdot P_D$$

i.e. D is pseudo-effective $\Rightarrow vol(D) \in \mathbb{Q}$

Proof. Choose m large enough such that $mD - \lceil mN \rceil = mP$. By definition the volume of D we have $vol_X(n \cdot D) = n^2 \cdot vol_X(D)$, so without loss of generality we may assume that D, N and P are integral divisors.

$$\lim_{m \to \infty} 2 \frac{h^0(X, \mathcal{O}_X(mD))}{m^2} = \lim_{m \to \infty} 2 \frac{h^0(X, \mathcal{O}_X(mP))}{m^2} = \lim_{m \to \infty} P \cdot P + 2 \frac{O(m)}{m^2} = P \cdot P \in \mathbb{Q}$$

□

Corollary 3. *The ring $R(X, D)$ is finitely generated if and only if P is semi-ample, i.e. $|lP|$ is free for some $l \gg 0$.*

Proof. See example 2.1.30 and theorem 2.3.15 in [PAG]. $\square$

Remark 4. *Zariski himself constructed an example of a divisor such that the canonical ring is not finitely generated. See [PAG] p.158 for details.*

3 Zariski decomposition in higher dimensions

Definition 2 (Cutkosky-Kawamata-Moriwaki-Decomposition)**.** *Let X be a smooth projective variety, a CKM decomposition of a divisor D is a birational modification $\mu : X' \to X$, together with an effective $\mathbb{Q}$-divisor N', such that*

i) $P' = \mu^* D - N'$ *is a nef divisor on* X'

ii) the natural maps

$$H^0(X', \mathcal{O}_{X'}(\mu^*(mD - \lceil mN \rceil))) \to H^0(X', \mathcal{O}_{X'}(\mu^*(mD)))$$

are bijective

Corollary 4. *If a divisor D admits a CKM decomposition then $vol_X(D) \in \mathbb{Q}$.*

Proof. The volume is a birational invariant i.e. $vol(D) = vol(\mu^* D)$, so the proof is similar to proof of Corollary 2. $\square$

Remark 5. *Cutkosky constructed examples of effective big divisors with irrational volume to show that a CKM-Decomposition on a smooth projective variety does not exist in general.*

4 Applications

Let again X be a smooth projective surface, D be a $\mathbb{R}$-divisor, we define

$$Null(D) := \{C | C \text{ is an irreducible curve with } D \cdot C = 0\}$$

If $D = P + N$ is its Zariski decomposition, then define

$$Neg(D) := \{C | C \text{ is an irreducible component of } N\}$$

Denote by $\mathcal{I}(X)$ the set of all irreducible curves with negative self intersection. For $C \in \mathcal{I}(X)$ define

$$C^{\geq 0} := \{D \in NS_{\mathbb{R}}(X) | D \cdot C \geq 0\}$$

and

$$C^{\perp} := \{D \in NS_{\mathbb{R}}(X) | D \cdot C = 0\}$$

Lemma 4. *The intersection of the nef cone $Nef(X)$ and the big cone $Big(X)$ is local polyhedral, i.e. for every $\mathbb{R}$-divisor $P \in Big(X) \cap Nef(X)$ there exists an open neighborhood U and curves $C_1, \ldots, C_q \in \mathcal{I}(X)$ such that*

$$U \cap V = U \cap (C_1^{\geq 0} \cap \cdots \cap C_q^{\geq})$$

Theorem 3 (Variation of the Zariski decomposition)**.** *Let X be a smooth projective surface. Then there is a locally finite decomposition of the big cone into rationally polyhedral subcones, such that in each subcone the support of the negative part of the Zariski decomposition of the divisors in the subcone is constant.*

Idea of the proof of Theorem 3. For a nef divisor P we define the chamber of P to be

$$\Sigma_P := \{D \in Big(X) | Neg(D) = Null(P)\}$$

Now one of the key observations is that $Big(X)$ is the union of the sets Σ_P, [BKS] lemma 1.4. The face of P is defined by

$$Face(P) = \bigcap_{C \in Null(P)} C^{\perp} \cap Nef(X)$$

Let $V^{\geq 0}(M)$ be the cone generated by the subset $M \subset NS(X)_{\mathbb{R}}$. By proposition 1.8 [BKS] we can extend the local polyhedral property of the face to the chamber Σ_P via

$$Big(X) \cap \Sigma_P = (Big(X) \cap Face(P)) + V^{\geq 0}(Null(P))$$

since by Lemma 4 $Big(X) \cap Face(P)$ is local polyhedral. By definition of Σ_P, the support of the negative part of the Zariski decomposition is constant on Σ_P, so it remains to show that this decomposition is locally finite. This follows from the lemma 4 below, since every big divisor has an open neighborhood in $Big(X)$ in the form of $D + Amp(X)$ for some big divisor D, and so only finitely many chambers Σ_P can meet the neighborhood . □

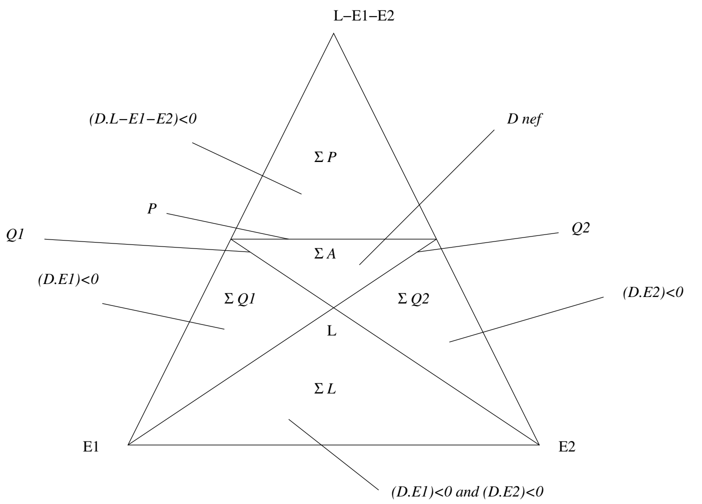
Example 4. *Take again the the projective space $\mathbb{P}^2$ blown up at two points P_1 and P_2, in this case we have curves with negative self intersection, two exceptional divisors E_1, E_2 and the strict transform L of the line through P_1 and P_2 i.e. $h - E_1 - E_2$. The intersection matrix of $\{L, E_1, E_2\}$ is*

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

For an effective divisor $D = aL + bE_1 + cE_2$ there are five possibilities for the Zariski decomposition

$$D = \begin{cases} (aL + bE_1 + cE_2) + 0 & \text{if } a \geq b, a \geq c, b + c \geq a \\ (aL + bE_1 + cE_2) + ((b - a)E_1 + (c - a)E_2) & \text{if } a \leq b, a \leq c \\ (aL + bE_1 + cE_2) + (b - a)E_1 & \text{if } c \leq a \leq b \\ (aL + bE_1 + cE_2) + (c - a)E_2 & \text{if } b \leq a \leq c \\ ((b + c)L + bE_1 + cE_2)) + (a - b - c)L & \text{if } b + c \leq a \end{cases}$$

where the positive part has been written first. The hyperplanes corresponding to L, E_1 and E_2 decompose the big cone into 5 parts on each of which the support of the negative part of Zariski decomposition remains constant (look at the list above) and so determine the chamber structure. In the picture below A is an arbitrary ample divisor, P, Q_1 and Q_2 are big and nef divisors in the nef boundary which are in the relative interiors of the indicated faces, such that the decomposition can be represented by the corresponding chambers $\Sigma_A, \Sigma_P, \Sigma_{Q_1}, \Sigma_{Q_2}$ and Σ_L.



Lemma 5. *If D is a big $\mathbb{R}$-divisor and A is an ample $\mathbb{R}$-divisor, then*

$$Neg(D + \lambda A) \subset Neg(D)$$

for all $\lambda \geq 0$

Proof. Take $\varepsilon_0 > 0$ from the proposition below, if $\lambda \leq \varepsilon_0$, then the statement follows from the proposition below. If $\lambda > \varepsilon_0$, then there exists a $n \in \mathbb{N}$ such that $\lambda \leq n\varepsilon_0$. Then $D + \lambda A = D + \varepsilon_0 A + (\lambda - \varepsilon_0)A$. The statement is true for $D + \varepsilon_0 A$(which is also big) so the proof can done by induction. □

Proposition 2. *Let D be a big divisor with Zariski decomposition $D = P + N$ and A an ample divisor, such that $N = \sum_{i=1}^r a_i C_i$ where C_i are integral curves, then there exists a positive number $\varepsilon_0 > 0$ and affine-linear functions $f_1, \ldots, f_r : \mathbb{R} \to \mathbb{R}$, such that for all $0 \leq \varepsilon \leq \varepsilon_0$ the Zariski decomposition of $D + \varepsilon A$ is*

$$\underbrace{P + \varepsilon A + \sum_{i=1}^{r} (a_i - f_i(\varepsilon))C_i}_{\textit{positive part}} + \sum_{i=1}^{r} f_i(\varepsilon)C_i$$

Proof. Let

$$P' = P + \varepsilon A + \sum_{i=1}^{r} (a_i - x_i)C_i$$

then $D + \varepsilon A$ has the Zariski decomposition

$$\underbrace{P'}_{\textit{positive part}} + \sum_{i=1}^{r} x_i C_i$$

if

i) $0 \leq x_i \leq a_i$ for all i

ii) $P' \cdot C_i = 0$ for all i

Let us explain this. First we should check that i) and ii) imply that P' is the positive part of the Zariski decomposition: Since $P+\varepsilon A$ is ample ii) implies that P' is nef, $N' = \sum_{i=1}^{r} x_i C_i$ is effective by i) and has by definition negative definite intersection matrix or is trivial. Now we will investigate, when conditions i) and ii) are satisfied. Let S be the intersection matrix of N, then $P' \cdot C_i = 0$ for all i is equivalent to

$$S \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} = \varepsilon \begin{pmatrix} AC_1 \\ \vdots \\ AC_r \end{pmatrix} + \begin{pmatrix} NC_1 \\ \vdots \\ NC_r \end{pmatrix}$$

Since S is negative definite, the system has a unique solution

$$\begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} = \varepsilon S^{-1} \begin{pmatrix} AC_1 \\ \vdots \\ AC_r \end{pmatrix} + \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix}$$

Therefore the x_i can be understood as affine linear functions f_i of ε. On the other hand, S has only positive or zero entries besides the diagonal ones, this forces all entries of S^{-1} to be negative or zero [BKS] theorem 4.1. Thus every f_i is in the form of $f_i(\varepsilon) = -b\varepsilon + a_i$ with $b \in \mathbb{R}_{\geq 0}$. Condition i) will be satisfied if we take ε_0 to be the smallest zero of the f_i's, this was to show! □

Corollary 5 (Continuity of the Zariski decomposition)**.** *If (D_n) is a sequence of big divisors converging in $NS_{\mathbb{R}}(X)$ to a big divisor D, $D_n = P_n + N_n$ and $D = P + N$ are the corresponding Zariski decompositions, then (P_n) will converge to P and (N_n) to N.*

Proof. [BKS] proposition 1.14. □

5 Appendix

5.1 Blowing up projective surfaces at a point

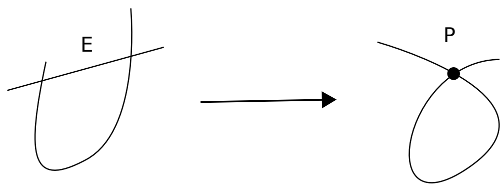
We discuss the basic properties of blow ups of points in the case of non singular projective surfaces.

Theorem 4. *Let X be a projective smooth surface. If X' is the blow up of X at some point $p \in X$, then there is a morphism of varieties $\pi : X' \to X$ such that $X' - \pi^{-1}(\{p\}) \cong X - \{p\}$ and $\pi^{-1}(\{p\}) \cong \mathbb{P}^1$. The exceptional divisor E corresponding to $\pi^{-1}(\{p\})$ has self intersection number -1. The canonical map $\pi^* : Pic(X) \to Pic(X')$ and $\mathbb{Z} \to Pic(X')$, $n \to En$ determine the intersection theory of X' by the following rules:*

- $Pic(X') \cong Pic(X) \times \mathbb{Z}$
- $E^2 = -1$
- $\pi^*(C) \cdot \pi^*(D) = C \cdot D$ *for* $C, D \in Pic(X)$
- $\pi^*(C) \cdot E = 0$ *for* $C \in Pic(X)$

- *Let $\pi_* : Pic(X') \to Pic(X)$ denote the canonical map, then for $C \in Pic(X)$ and $D \in Pic(X')$ we have*
$$C \cdot \pi_*(D) = \pi^*(C) \cdot D$$

Proof. See proposition 3.2 in [Har]. □



5.2 The Riemann-Roch Theorem

In this section we discuss two Riemann-Roch type theorems.

Theorem 5 (Asymptotic Riemann-Roch Theorem)**.** *Let D be a nef divisor on a smooth projective variety of dimension m. Then*
$$h^0(X, \mathcal{O}(nD)) = \frac{D^m}{m!} \cdot n^m + O(n^{m-1})$$

Proof. See for example Lazarsfeld, [PAG], p. 69 □

Theorem 6 (Riemann-Roch theorem for surfaces)**.** *Let X be a smooth projective surface, D an effective divisor and K_X the canonical divisor, then*
$$h^0(X.\mathcal{O}_X(mD)) - h^1(X.\mathcal{O}_X(mD)) + h^2(X.\mathcal{O}_X(mD)) = \frac{1}{2} D \cdot D m^2 - D \cdot K m + \chi(X, \mathcal{O}_X(mD))$$

Proof. See theorem 1.6 on p. 363 [Har], note that $p_a = \chi(\mathcal{O}_X) - 1$. □

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