ASYMPTOTIC INVARIANTS OF LINEAR SERIES

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ABSTRACT. These are the notes for my talk about the Nakayama-Zariski decomposition. Everything is contained in [Nak04].

CONTENTS

1. ASYMPTOTIC ORDER OF VANISHING

In this section, let X be a smooth projective complex variety and let $\Gamma \subset X$ be a prime divisor.

A *divisor* B on X is a divisor with coefficients in $\mathbb{R}$. We write $B \geq 0$ for an effective divisor. Numerical equivalence of divisors is denoted by $\equiv_{num}$. Linear equivalence over $\mathbb{K} = \mathbb{Q}, \mathbb{R}$ is denoted by $\sim_{\mathbb{K}}$. The coefficient of Γ in B is denoted by $mult_{\Gamma}(B)$. The coefficient of B in a general element of a non-empty linear system $|V|$ is denoted by $mult_{\Gamma}|V|$. The big cone of X is denoted by $Big(X)$. Its closure $PE(X) = \overline{Big(X)}$ is the pseudoeffective cone.

Definition 1.1. *Let B be a big divisor on X. The* asymptotic order of vanishing of B along Γ *is*

$$\sigma_{\Gamma}(B) = inf\{mult_{\Gamma}(B') : B' \geq 0 \text{ and } B' \equiv_{num} B\}.$$

Proposition 1.2. *Let B, B_i be big divisors on X, and $c \in \mathbb{R}_{>0}$. Then the following statements hold true:*

1. $\sigma_{\Gamma}(c \cdot B) = c \cdot \sigma_{\Gamma}(B)$ $\forall c \in \mathbb{R}_{>0}$ *(homogeneity)*
2. $\sigma_{\Gamma}(B_1 + B_2) \leq \sigma_{\Gamma}(B_1) + \sigma_{\Gamma}(B_2)$ *(subadditivity)*
3. $\sigma_{\Gamma}(A) = 0$ *for any ample divisor A on X. In particular* $\sigma_{\Gamma}(B + A) \leq \sigma_{\Gamma}(B)$.
4. $\sigma_{\Gamma} : Big(X) \to \mathbb{R}_{\geq 0}$ *is a continuous function, where $Big(X)$ is the big cone.*

Proof. The statements 1 and 2 are easy. 3 follows from the fact that an ample integral divisor is basepoint-free and an ample divisor is a linear combination with positive coefficients of ample integral divisors.

In order to prove 4 we choose an ample divisor A on X. There exists a $\delta > 0$ such that $B - \delta A \sim_{\mathbb{R}} N \geq 0$ for some effective divisor N. This implies that for

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$\epsilon > 0$,

$$\begin{aligned}\sigma_\Gamma(B) &\leq (1+\epsilon)\sigma_\Gamma(B) \\ &= \sigma_\Gamma(B+\epsilon B) \\ &= \sigma_\Gamma(B+\epsilon\delta A+\epsilon N) \\ &\leq \sigma_\Gamma(B+\epsilon\delta A)+\epsilon\cdot mult_\Gamma(N) \\ &\leq \sigma_\Gamma(B)+\epsilon\cdot mult_\Gamma(N)\end{aligned}$$

Consequently, $\sigma_\Gamma(B) = \lim_{\epsilon\to 0,\epsilon>0}\sigma_\Gamma(B+\epsilon A)$. In a similar vein one can show that $\sigma_\Gamma(B) = \lim_{\epsilon\to 0,\epsilon<0}\sigma_\Gamma(B+\epsilon A)$. The continuity is now an easy consequence of 3. □

Corollary 1.3. *Let B be a big integral divisor. Then*

$$\sigma_\Gamma(B) = \lim_{m\to\infty}\frac{1}{m}mult_\Gamma|mB|$$

where m runs over all integers such that $|mB|$ is a non-empty linear system.

Sketch of proof. The right hand side exists and equals

$$\sigma_\Gamma(B)_{\mathbb{Q}} := inf\{mult_\Gamma B' :\ \text{B}'\ \mathbb{Q}\text{-divisor},\ B'\geq 0,\ B'\sim_{\mathbb{Q}} B\}.$$

Let $\sigma_\Gamma(B)_{\mathbb{R}}$ be defined similarly, only replacing $\mathbb{Q}$-divisors and linear equivalence over $\mathbb{Q}$ by divisors and linear equivalence over $\mathbb{R}$. Then $\sigma_\Gamma(B)_{\mathbb{Q}} = \sigma_\Gamma(B)_{\mathbb{R}}$ holds true, since the rational points of a rational polyhedron in a finite dimensional real vector space defined over $\mathbb{Q}$ form a dense subset of the polyhedron.

It remains to show that $\sigma_\Gamma(B) = \sigma_\Gamma(B)_{\mathbb{R}}$. The inequality $\sigma_\Gamma(B)\leq\sigma_\Gamma(B)_{\mathbb{R}}$ holds since linear equivalence implies numerical equivalence. In order to show the reverse inequality, we remark that the proof of Proposition 1.2 also applies to $\sigma_\Gamma(B)_{\mathbb{R}}$. In particular, $\sigma_\Gamma(B)_{\mathbb{R}} = lim_{\epsilon\to 0}\sigma_\Gamma(B+\epsilon A)_{\mathbb{R}}$ for some ample divisor A. Thus it suffices to show that $\sigma_\Gamma(B)\geq\sigma_\Gamma(B+\epsilon A)_{\mathbb{R}}$ for an arbitrary ample divisor A. This last assertion is true since ampleness is a numerical property. □

Corollary 1.4. *Suppose that X is a surface. Let $B = P+N$ be the Zariski-decomposition of a big divisor. Then $\sigma_\Gamma(B) = mult_\Gamma(N)$.*

Proof. By continuity of the Zariski-decomposition and 1.2.4 we may assume that B is a rational divisor. By homogeneity of the Zariski-decomposition and 1.2.2 we may assume that all involved divisors are integral. Recall that the positive part of the Zariski-decomposition carries all sections, i.e., $|mB| = |mP|+mN$ for $m\in\mathbb{N}$. The nefness of P and 1.2.4 imply that $\sigma_\Gamma(P) = 0$. The corollary follows with

$$\begin{aligned}\sigma_\Gamma(B) &= \lim_{m\to\infty}\frac{1}{m}mult_\Gamma|mB| \\ &= \lim_{m\to\infty}\frac{1}{m}\big(mult_\Gamma(|mP|)\big)+mult_\Gamma N \\ &= \sigma_\Gamma(P)+mult_\Gamma N \\ &= mult_\Gamma N.\end{aligned}$$

□

Proposition 1.5. *Let A be an ample divisor on X. Then the asymptotic order of vanishing $\sigma_\Gamma : Big(X)\to\mathbb{R}_{\geq 0}$ extends to the pseudoeffective cone $PE(X)$ by*

$$\sigma_\Gamma(D) := \lim_{\epsilon\to 0,\epsilon>0}\sigma_\Gamma(D+\epsilon A),\quad D\in PE(X).$$

The function $\sigma_\Gamma : PE(X) \to \mathbb{R}_{\geq 0}$ does not depend on the choice of A.

Remark 1.6. The function $\sigma_\Gamma : PE(X) \to \mathbb{R}_{\geq 0}$ is not necessarily continuous.

Proof. By 1.2.3 the limit exists in $[0, \infty]$. By Lemma 1.7, $D + \epsilon A - \sigma_\Gamma(D + \epsilon A) \cdot \Gamma$ is big. As a consequence,

$$(D + \epsilon A - \sigma_\Gamma(D + \epsilon A) \cdot \Gamma) \cdot A^{n-1} \geq 0$$

and

$$\sigma_\Gamma(D + \epsilon A) \leq \frac{(D + \epsilon A) \cdot A^{n-1}}{\Gamma \cdot A^{n-1}} \leq C < \infty$$

is bounded for $\epsilon \to 0$.

Let A' be another ample divisor. Then $A - \delta A'$ is ample for some $\delta > 0$ and $\sigma_\Gamma(D + \epsilon A) \leq \sigma_\Gamma(D + \epsilon\delta A')$. Consequently, $\lim_{\epsilon \to 0} \sigma_\Gamma(D + \epsilon A) \leq \lim_{\epsilon \to 0} \sigma_\Gamma(D + \epsilon A')$ and the reverse inequality holds by symmetry. □

Lemma 1.7. *Let B be a big divisor and let Γ_i be finitely many pairwise distinct prime divisors. Then $B - \sum_i \sigma_{\Gamma_i}(B) \cdot \Gamma_i$ is also big.*

Proof. There exist an ample divisor A and an effective divisor $E \geq 0$ such that $B \sim_{\mathbb{R}} A + E$. We calculate $\sigma_{\Gamma_i}(B) \leq \sigma_{\Gamma_i}(A) + \sigma_{\Gamma_i}(E) \leq mult_{\Gamma_i}(E)$. Thus $B - \sum_i \sigma_{\Gamma_i}(B) \cdot \Gamma_i \sim_{\mathbb{R}} A + (E - \sum_i \sigma_{\Gamma_i}(B) \cdot \Gamma)$ is big. □

Proposition 1.8. *Let $D \in PE(X)$ be a pseudoeffective divisor on X. Then there exist only finitely many prime divisors $\Gamma \subset X$ such that $\sigma_\Gamma(D) > 0$.*

Proof. Choose an ample divisor A. If $\sigma_\Gamma(D) > 0$, then $\sigma_\Gamma(D + \epsilon A) > 0$ for *all* sufficiently small $\epsilon > 0$. In particular, it suffices to show that for any *big* divisor B there exist at most $\rho(X)$ prime divisors Γ such that $\sigma_\Gamma(B) > 0$. This can be seen as follows: Let Γ_i, $1 \leq i \leq p$, be finitely many pairwise distinct prime divisors such that $\sigma_{\Gamma_i}(B) > 0$. By Lemma 1.7, $B - \sum_i x_i \Gamma_i$ is big for $0 < x_i < \sigma_{\Gamma_i}(B)$. The definition of σ_{Γ_j} immediately implies that

$$\sigma_{\Gamma_j}(B - \sum_i x_i \Gamma_i) = \sigma_{\Gamma_j}(B) - x_j.$$

Since σ_{Γ_j} is a numerical invariant, the prime divisors Γ_i are linearly independent in the Neron-Severi-space. Thus the number of prime divisors Γ_i is bounded by the Picard number, i.e., $p \leq \rho(X)$. □

Definition 1.9. *The* Nakayama-Zariski decomposition *of a pseudoeffective divisor $D \in PE(X)$ is given as*

$$D = N_\sigma(D) + P_\sigma(D)$$

where $N_\sigma(D) = \sum_\Gamma \sigma_\Gamma(D) \cdot \Gamma$ and $P_\sigma(D) = D - N_\sigma(D)$. $N_\sigma(D)$ (resp. $P_\sigma(D)$) is called the negative part *(resp., the* positive part*).*

2. THE MAIN THEOREM

The aim of this section is to give an overview over the proof of the following result.

Theorem 2.1. *Let X be a smooth projective variety and let $D \in PE(X)$ be a pseudoeffective divisor. Suppose that $D \not\equiv_{num} 0$ and that $N_\sigma(D) = 0$. Then there exists an ample integral divisor A and a constant $\beta > 0$ such that for all $m \gg 0$,*

$$h^0(X, \mathcal{O}_X(\lfloor mD \rfloor + A)) \geq \beta \cdot m$$

In the remainder of this section we prove Theorem 2.1. Let X be a smooth projective variety.

Lemma 2.2 ([Nak04], Ch. 6)**.** *Let Δ be an effective divisor on X, $L \in Pic(X)$ and let $S \subset X$ be either a point or a smooth curve. Suppose that*

- *(X, Δ) is log terminal along S, and*
- *Let $\rho_S : Bl_S(X) \to X$ be the blowing-up along S with exceptional divisor $E_X \subset Bl_S(X)$. Then $\rho_S^*(L - (K_X + \Delta)) - (n - dim(S))E_S$ is ample.*

Then the restriction map $H^0(X, L) \to H^0(S, L|_S)$ is surjective.

Recall that the stable base locus of a $\mathbb{Q}$-divisor B is the closed subset in X defined by

$$\mathbb{B}(B) = \bigcap_{\substack{B' \geq 0 \\ B' \sim_{\mathbb{Q}} B}} Supp(B').$$

Definition 2.3. *The restricted base locus of a pseudoeffective divisor $D \in PE(X)$ is*

$$\mathbb{B}_-(D) = \bigcup_{\substack{A \text{ ample} \\ D+A\ \mathbb{Q}\text{-div.}}} \mathbb{B}(D + A)$$

Notation 2.4. For any point $x \in X$, let $\rho_x : B_x(X) \to X$ be the blowing-up of $x \in X$ with exceptional divisor $E_x \subset B_x(X)$. For any pseudoeffective divisor $D \in PE(X)$, let

$$\sigma_x(D) = \sigma_{E_x}(\rho_x^*(D)).$$

Proposition 2.5. *Let $D \in PE(X)$ be a pseudoeffective divisor on X. Then $\mathbb{B}_-(D) = \{x \in X : \sigma_x(D) > 0\}$.*

Proof. For simplicity, assume that D is a $\mathbb{Q}$-divisor.

Suppose that $x \notin \mathbb{B}_-(D)$, and let A be an ample $\mathbb{Q}$-divisor. Then $x \notin \mathbb{B}(D + A)$ so that $\sigma_x(D + A) = 0$. Let A' be an ample divisor on $B_x(X)$. Then $\left(\rho_x^*(D) + A'\right) - \rho_x^*(D + \epsilon \cdot A)$ is ample for sufficiently small $\epsilon > 0$. In particular, $\sigma_{E_x}(\rho_x^*(D) + A') \leq \sigma_{E_x}(\rho_x^*(D + \epsilon A)) = 0$ vanishes. Consequently, $\sigma_x(D) = 0$.

Suppose conversely that $\sigma_x(D) = 0$ and let A be an ample $\mathbb{Q}$-divisor. Then

$$\rho_x((m - 1)A - K_X) - n \cdot E_x$$

is an ample divisor on $B_x(X)$ for some sufficiently divisible $m \gg 0$. In order to prove that $x \notin \mathbb{B}(D + A)$ it suffices by Lemma 2.2 to show the existence of an effective divisor Δ on X satisfying

- (X, Δ) is log terminal at $x \in X$,
- $(m - 1)A - K_X \equiv_{num} m(D + A) - (K_X + \Delta)$.

The second condition is equivalent to $\Delta \equiv mD + A$. The existence of such a Δ is due to $\sigma_x(mD + A) = 0$. □

Corollary 2.6. *Under the assumptions of Theorem 2.1, $\mathbb{B}_-(D)$ is a countable union of closed subvarieties in X of codimension at least* 2*.*

Proof. $\mathbb{B}_-(D)$ is a countable union of closed subvarieties since in the definition of $\mathbb{B}_-(D)$ one can replace the union over all A by the union of a sequence of ample divisors A_n converging to 0. If $\Gamma \subset X$ is a prime divisor contained in $\mathbb{B}_-(D)$, then $\sigma_x(D) > 0$ for any point $x \in X$. As a consequence $\sigma_\Gamma(D) > 0$, which contradicts the assumption $N_\sigma(D) = 0$. □

Proof of Theorem 2.1. For simplicity, assume that D is integral. If $C \subset X$ is a curve given by the intersection of very general hyperplane sections, then $C \cap \mathbb{B}_-(D) = \emptyset$ by Corollary 2.6. Further, $(C \cdot D) > 0$ since $D \not\equiv_{num} 0$. If A is a sufficiently ample integral divisor, then

$$\rho_C^*(A - K_X) - (n - 1)E_C$$

is an ample divisor on the blowing-up $B_C(X)$ of X along C (with exceptional divisor E_C). In order to apply Lemma 2.2 we seek to write $A - K_X \sim_{\mathbb{Q}} (mD + 2A) - (K_X + \Delta)$, where $(X, \Delta \geq 0)$ is log terminal along $C \subset X$. The first condition is equivalent to $\Delta \sim_{\mathbb{Q}} mD + A$. The linear system $|N(mD + 2A)|$ is basepoint-free along C for some $N \gg 0$. In particular, any general member $F \in |N(mD + 2A)|$ meets C transversally. This implies that $\Delta := \frac{1}{N}F \sim_{\mathbb{Q}} mD + 2A$ is log terminal along C and satisfies the condition of Lemma 2.2. In particular, the restriction map

$$H^0(X, mD + 2A) \to H^0(C, (mD + 2A)|_C)$$

is surjective for $m \gg 0$. As $h^0(C, (mD + A)|_C)$ grows like $(C \cdot D)m$, the assertion follows. $\square$

REFERENCES

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