1 Introduction

1.1 Motivation

Our goal is to understand Lazić’s proof of the following theorem.

**Theorem 1.1.** Let $X$ be a smooth projective variety. Then the canonical ring

$$R(X, K_X) := \bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(mK_X))$$

is a finitely generated $\mathbb{C}$-algebra (“is finitely generated”, for short).

We want to give some reasons why this is an interesting result.

- When classifying projective varieties, one looks for a natural embedding in projective space. Such an embedding exists whenever we have a variety whose canonical divisor is ample. So we ask, given a variety $X$, is $X$ birational to a variety with ample canonical divisor? For this to hold, $X$ needs to be of general type, meaning that $K_X$ is big. Conversely, given a smooth projective variety $X$ of general type with finitely generated canonical ring, we can consider the variety

$$X_{\text{can}} := \text{Proj } R(X, K_X),$$

called the *canonical model* of $X$. One can show that $X_{\text{can}}$ is a normal variety with canonical singularities which is birational to $X$ and whose canonical divisor $K_{X_{\text{can}}}$ is ample. It is also possible to describe explicitly a birational map $X \dashrightarrow X_{\text{can}}$: Choose a positive integer $r$ such that the $r$-th Veronese subring $R(X, rK_X)$ is generated by $H^0(X, rK_X)$. Then the complete linear system $|rK_X|$ defines a rational map

$$X \dashrightarrow \mathbb{P}(H^0(X, rK_X)^*)$$

which is birational onto its image, and the image is precisely $X_{\text{can}}$.

- For any variety, the canonical ring is one of two rings naturally attached to it (the Cox ring is the other). Therefore it is natural to ask about its properties.

- In the minimal model program, when one encounters a contraction $f : X \rightarrow Y$ whose exceptional locus has codimension $\geq 2$, it is not possible to contract further because $Y$ is too singular ($K_Y$ is not $\mathbb{Q}$-Cartier). Instead, one would like to replace $f$ by its *flip*, which is a variety $X^+$ together with a morphism $f^+ : X^+ \rightarrow Y$ such that $K_{X^+}$ is $f^+$-ample. The existence of flips was a big problem in minimal model
theory, but it is easy to see that it is equivalent to the finite generation of the relative canonical ring

\[ \bigoplus_{m \geq 0} f_* \mathcal{O}_X(mK_X), \]

which in turn follows from Theorem 1.1.

1.2 Main statement

In fact, Lazić proves a theorem slightly different from Theorem 1.1:

Theorem A. Let \( X \) be a smooth projective variety of dimension \( n \). Let \( B_1, \ldots, B_k \) be \( \mathbb{Q} \)-divisors on \( X \) such that \( [B_i] = 0 \) for all \( i \), and such that the support of \( \sum_{i=1}^k B_i \) has simple normal crossings. Let \( A \) be an ample \( \mathbb{Q} \)-divisor on \( X \), and denote \( D_i = K_X + A + B_i \) for every \( i \).

Then the adjoint ring

\[ R(X; D_1, \ldots, D_k) = \bigoplus_{(m_1, \ldots, m_k) \in \mathbb{N}^k} H^0(X, \mathcal{O}_X([\sum m_i D_i])) \]

is finitely generated.

Let us indicate briefly how to deduce Theorem 1.1 from Theorem A. If \( X \) is a smooth projective variety with \( \kappa(X, K_X) \geq 0 \), we have the Iitaka fibration

\[ X - \varphi_{[mK_X]} \rightarrow \mathbb{P}(H^0(X, mK_X)^*) \]

\[ \downarrow \]

\[ Y \]

defined by the sections of a suitable multiple of \( K_X \). Now Fujino and Mori have shown that there exists an effective divisor \( \Delta \) on \( Y \) such that \((Y, \Delta)\) is klt and \( R(X, K_X) \) and \( R(Y, K_Y + \Delta) \) have isomorphic Veronese subrings. So the former is finitely generated if and only if the latter is. But note that

\[ \kappa(Y, K_Y + \Delta) = \kappa(X, K_X) = \dim Y, \]

which means that \( K_Y + \Delta \) is big. Thus we may write \( K_Y + \Delta \sim_{\mathbb{Q}} A + B \) with \( A \) ample and \( B \geq 0 \). For sufficiently small rational \( \varepsilon > 0 \), set \( \Delta' = \varepsilon A + (\Delta + \varepsilon B) \). Then \( K_Y + \Delta' \sim_{\mathbb{Q}} (1 + \varepsilon)(K_Y + \Delta) \), so \( R(Y, K_Y + \Delta) \) and \( R(Y, K_Y + \Delta') \) have isomorphic Veronese subrings, hence it suffices to prove that \( R(Y, K_Y + \Delta') \) is finitely generated. Since \((Y, \Delta')\) is klt, this follows from the \( k = 1 \) case of Theorem A (after passing to a log resolution).
1.3 Some definitions

Before we can give an outline of the proof of Theorem A, we need some definitions.

**Definition 1.2.** Let \((X, S + \sum_{i=1}^{p} S_i)\) be a log smooth projective pair, where \(S\) and all \(S_i\) are distinct prime divisors, let \(V = \sum_{i=1}^{p} \mathbb{R}S_i \subseteq \text{Div}_\mathbb{R}(X)\), and let \(A\) be a \(\mathbb{Q}\)-divisor on \(X\). We define

\[
L(V) = \{ B = \sum b_i S_i \in V \mid 0 \leq b_i \leq 1 \text{ for all } i\},
\]
\[
E_A(V) = \{ B \in L(V) \mid |K_X + A + B|_\mathbb{R} \neq \emptyset\},
\]
\[
B^S_A(V) = \{ B \in L(V) \mid S \not\subseteq B(K_X + S + A + B)\}.
\]

These definitions are best remembered by noting that:

- \(L(V)\) is defined by the condition that a certain pair is log canonical,
- \(E_A(V)\) by the condition that a certain divisor is effective, and
- \(B^S_A(V)\) by the condition that a certain stable base locus does not contain \(S\).

Also note that in the definition of \(B^S_A(V)\), one considers \(K_X + S + A + B\) instead of \(K_X + A + B\) because one would like to apply the adjunction formula. For example, if \(B \in B^S_A(V)\) then \(B|_S \in E_{A|_S}(W)\), where \(W = \sum_{i=1}^{p} \mathbb{R}S_i|_S \subseteq \text{Div}_\mathbb{R}(S)\).

It is clear that \(L(V)\) is a rational polytope (with respect to the canonical basis of \(V\)). Indeed, it is simply given by the “hypercube” \([0, 1]^p\). On the other hand, \(E_A(V)\) and \(B^S_A(V)\) are a priori only bounded convex sets. A precise analysis of their structure is one of the main ingredients to the proof.
1.4 Outline of the proof

The proof of Theorem A is by induction on the dimension of $X$. As part of the induction, the following theorems are additionally proven.

**Theorem B.** Let $(X, \sum_{i=1}^{p} S_i)$ be a log smooth projective pair of dimension $n$, where $S_1, \ldots, S_p$ are distinct prime divisors. Let $V = \sum_{i=1}^{p} R S_i \subseteq \text{Div}_R(X)$, and let $A$ be an ample $\mathbb{Q}$-divisor on $X$.

Then $E_A(V)$ is a rational polytope.

**Theorem 1.3.** Let $(X, S + \sum_{i=1}^{p} S_i)$ be a log smooth projective pair of dimension $n$, where $S$ and all $S_i$ are distinct prime divisors. Let $V = \sum_{i=1}^{p} R S_i \subseteq \text{Div}_R(X)$ and let $A$ be an ample $\mathbb{Q}$-divisor on $X$. Then $B_A^S(V)$ is a rational polytope, and

$$B_A^S(V) = \{ B \in \mathcal{L}(V) \mid \sigma_S(K_X + S + A + B) = 0 \}.$$  

The precise structure of the induction is the following.

- Theorems $A_{n-1}$ and $B_{n-1}$ imply Theorem 1.3 (see Section 3).
- Theorems $A_{n-1}$, $B_{n-1}$ and 1.3 imply Theorem $B_n$ (see Section 4).
- Theorems $A_{n-1}$ and $B_n$ imply Theorem $A_n$ (see Section 5).

Here and elsewhere, “Theorem $A_n$” means “Theorem A in the case dim $X = n$”, and so on.

2 The lifting lemma

The following theorem, which is known as the “lifting lemma”, is crucial to the proof. It was originally proved by Hacon and McKernan. In a nutshell, it gives a sufficient condition for sections of an adjoint line bundle $K_S + \cdots$ on a divisor $S \subset X$ to be liftable to sections of $K_X + S + \cdots$. The condition says that the sections we would like to lift need to vanish along certain divisors (to certain orders).

**Theorem 2.1.** Let $(X, S + \sum_{i=1}^{p} S_i)$ be a log smooth projective pair, where $S$ and all $S_i$ are distinct prime divisors. Let $V = \sum_{i=1}^{p} R S_i \subseteq \text{Div}_R(X)$, and let $B \in \mathcal{L}(V)$ be a $\mathbb{Q}$-divisor such that $(S, B|_S)$ is canonical. Let $A$ be an ample $\mathbb{Q}$-divisor on $X$ and denote $\Delta = S + A + B$. Let $C \geq 0$ be a $\mathbb{Q}$-divisor on $S$, and let $m$ be a positive integer such that $mA$, $mB$ and $mC$ are integral.

Assume that there exists a positive integer $q \gg 0$ such that $qA$ is very ample, $S \not\subseteq Bs|qm(K_X + \Delta + \frac{1}{m} A)|$ and

$$C \leq \max \left\{ B|_S - \frac{1}{qm} \text{Fix}|qm(K_X + \Delta + \frac{1}{m} A)|_S, 0 \right\}.$$
\((The\ maximun\ is\ taken\ component-wise.\)\ Then
\[
|\text{m}(K_S + A|S + C)| + \text{m}(B|S - C) \leq |\text{m}(K_X + \Delta)|_S.
\]
In particular, if \(|\text{m}(K_S + A|S + C)| \neq \emptyset\), then \(|\text{m}(K_X + \Delta)|_S \neq \emptyset\), and
\[
\text{Fix} |\text{m}(K_S + A|S + C)| + \text{m}(B|S - C) \geq \text{Fix} |\text{m}(K_X + \Delta)|_S \geq m \text{Fix}_S(K_X + \Delta).
\]

The following is an immediate consequence of the lifting lemma. Here we see very clearly what is happening: If a divisor \(D \in |\text{m}(K_S + A|S + B|S)|\) is liftable to \(|\text{m}(K_X + S + A + B)|\), then \(D \geq \text{Fix} |\text{m}(K_X + S + A + B)|_S\) by definition. In particular, \(D \geq m(B|S - \Phi_m)\). The lifting lemma says this necessary condition is also sufficient.

**Corollary 2.2.** Let \((X, S + \sum_{i=1}^{p} S_i)\) be a log smooth projective pair, where \(S\) and all \(S_i\) are distinct prime divisors. Let \(V = \sum_{i=1}^{p} \mathbb{R}S_i \subseteq \text{Div}_\mathbb{R}(X)\) and let \(B \in \mathcal{L}(V)\) be a \(\mathbb{Q}\)-divisor such that \((S, B|S)\) is canonical. Let \(A\) be an ample \(\mathbb{Q}\)-divisor on \(X\) and denote \(\Delta = S + A + B\). Let \(m\) be a positive integer such that \(mA\) and \(mB\) are integral, and such that \(S \not\subseteq Bs|m(K_X + \Delta)|\). Denote \(\Phi_m = \max\{B|S - \frac{1}{m} \text{Fix} |m(K_X + \Delta)|_S, 0\}\).

Then
\[
|\text{m}(K_S + A|S + \Phi_m)| + m(B|S - \Phi_m) = |\text{m}(K_X + \Delta)|_S.
\]

### 3 Proof of Theorem \([1.3]_t\)

Let \(|| \cdot ||\) be any norm on \(\mathbb{R}^n\). In the applications, \(|| \cdot ||\) will mostly be the sup-norm. This has the advantage that a closed ball of rational radius around a rational point is a rational polytope.

The following result is a very simple example of Diophantine approximation.

**Lemma 3.1.** Let \(x \in \mathbb{R}^n\) be a point, and fix a real number \(\varepsilon > 0\). Then there are finitely many points \(x_i \in \mathbb{R}^n\) and positive integers \(k_i\) such that \(k_ix_i\) are integral, \(||x - x_i|| < \varepsilon/k_i\), and \(x\) is a convex combination of the \(x_i\).

The next lemma gives a criterion for a set to be a rational polytope.

**Lemma 3.2.** Let \(\mathcal{P} \subset \mathbb{R}^n\) be a bounded convex set. Then \(\mathcal{P}\) is a rational polytope if and only if there is a constant \(\varepsilon > 0\) such that for all \(w \in \mathcal{P}\), \(v \in \mathbb{Q}^n\), and \(\ell \in \mathbb{N}^+\) with \(\ell v\) integral and \(||v - w|| < \varepsilon/\ell\), we have \(v \in \mathcal{P}\).

**Setup 3.3.** Let \((X, S + \sum_{i=1}^{p} S_i)\) be a log smooth projective pair of dimension \(n\), where \(S\) and all \(S_i\) are distinct prime divisors. Let \(V = \sum_{i=1}^{p} \mathbb{R}S_i \subseteq \text{Div}_\mathbb{R}(X)\), let \(A\) be an ample \(\mathbb{Q}\)-divisor on \(X\), and let \(W \subseteq \text{Div}_\mathbb{R}(S)\) be the
subspace spanned by the components of $\sum S_i$. For $\mathbb{Q}$-divisors $E \in \mathcal{E}_{A|S}(W)$ and $B \in \mathcal{B}_S^A(V)$, let

$$F(E) = \text{Fix}(K_S + A|S + E) \quad \text{and} \quad F_S(B) = \text{Fix}_S(K_X + S + A + B).$$

Denote

$$\Phi_m(B) = \max \{ B|S - \frac{1}{m} \text{Fix} m(K_X + S + A + B)|S, 0 \}$$

for every sufficiently divisible positive integer $m$, and let

$$\Phi(B) = \max \{ B|S - F_S(B), 0 \}.$$ 

Note that $\Phi(B) = \limsup \Phi_m(B)$.

**Theorem 3.4.** Assume Theorem $[A_{n-1}]$ and Theorem $[B_{n-1}]$, and let the assumptions of Setup $[3.3]$ hold. Let $G$ be a rational polytope contained in the interior of $\mathcal{L}(V)$, and assume that $(S,G|S)$ is terminal for every $G \in G$. Denote $\mathcal{P} = G \cap \mathcal{B}_S^A(V)$. Then:

1. $\mathcal{P}$ is a rational polytope,
2. $\Phi$ extends to a rational piecewise affine function on $\mathcal{P}$, and there exists a positive integer $\ell$ such that $\Phi(P) = \Phi_m(P)$ for every $P \in \mathcal{P}$ and every positive integer $m$ such that $mP/\ell$ is integral.

**Corollary 3.5.** Theorem $[A_{n-1}]$ and Theorem $[B_{n-1}]$ imply Theorem $[1.3]$.

## 4 Proof of Theorem $[B_n]$

**Lemma 4.1.** Let $(X,B)$ be a log smooth projective pair, where $B$ is a $\mathbb{Q}$-divisor such that $|B| = 0$, and let $A$ be a nef and big $\mathbb{Q}$-divisor.

If $K_X + A + B$ is numerically equivalent to an effective $\mathbb{R}$-divisor, then it is also linearly equivalent to an effective $\mathbb{Q}$-divisor.

**Sketch of proof.** An application of Kawamata-Viehweg vanishing tells us that $h^0$ of certain divisors equals their Euler characteristic. Then use the fact that the Euler characteristic is a numerical invariant.

The next lemma is in some sense complementary to Lemma 4.1. Both lemmas put together say that if $(X,\Delta)$ is klt, $\Delta$ is big, and $K_X + \Delta$ is pseudoeffective, then $K_X + \Delta$ is ($\mathbb{R}$-linearly) effective.

**Lemma 4.2.** Assume Theorems $A_{n-1}$ and $B_{n-1}$.

Let $(X,B)$ be a log smooth projective pair, where $B$ is an $\mathbb{R}$-divisor such that $|B| = 0$. Let $A$ be an ample $\mathbb{Q}$-divisor on $X$, and assume that $K_X + A + B$ is a pseudo-effective divisor such that $K_X + A + B \neq N_\sigma(K_X + A + B)$.

Then there exists an $\mathbb{R}$-divisor $F \geq 0$ such that $K_X + A + B \sim_\mathbb{R} F$. 

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Proof. Set $\Delta = A + B$. By the assumption $K_X + \Delta \not\equiv N_\sigma(K_X + \Delta)$, there is a number $k \in \mathbb{N}^+$ such that $kA$ is integral and

$$h^0(X, [m(K_X + \Delta)] + kA) \to \infty$$

as $m \to \infty$. In particular, we find an $m \in \mathbb{N}^+$ with

$$h^0(X, [mk(K_X + \Delta)] + kA) > \left(\frac{n + nk}{n}\right).$$

Since the right-hand side is the number of conditions that a given divisor has multiplicity $> nk$ at a certain point fixed in advance, we obtain an effective divisor $G \sim R mk(K_X + \Delta) + kA$ with $\text{mult}_x G > nk$, for some $x \notin \text{Supp} N_\sigma(K_X + \Delta)$.

Set $D := \frac{1}{mk}G$ and consider a log resolution $f : Y \to X$ of $(X, B + D)$ constructed by first blowing up $x$, giving an exceptional divisor $P \subset Y$. For any $0 \leq t \leq m$,

$$K_Y + C_t = f^*(K_X + B + tD) + E_t,$$

where $C_t, E_t$ are effective and do not have any common components. Now define

$$B_t := \max\{C_t - N_\sigma(K_Y + f^*A_t + C_t), 0\},$$

where $A_t := \left(1 - \frac{t}{m}\right)A$. We make a few observations:

1) $N_\sigma(K_Y + f^*A_t + C_t) = \left(1 + t\right)N_\sigma(f^*(K_X + \Delta)) + E_t$, so $B_t$ is continuous as a function of $t$.

2) $\min\{B_t, N_\sigma(K_Y + f^*A_t + B_t)\} = 0$.

3) $\lfloor B_0 \rfloor = 0$, but $\text{mult}_P B_m > 1$.

By 1) and 3), there is a minimal $0 < \lambda < m$ such that $\text{mult}_S B_\lambda = 1$ for some prime divisor $S$. Then by 2), $\sigma_S(K_Y + f^*A_\lambda + B_\lambda) = 0$. Now Theorem 1.3 told us that

$$S \notin \text{B}(K_Y + S + f^*A_\lambda + (B_\lambda - S)),$$

in particular,

$$K_Y + f^*A_\lambda + B_\lambda \sim R F' \geq 0$$

for some effective divisor $F'$. (Here we are actually cheating a little because $f^*A_\lambda$ is not ample, as required in Theorem 1.3, but only big and nef. In reality one needs to subtract a small effective exceptional divisor from $f^*A_\lambda$ to make things work.) The latter linear equivalence may be pushed down to $X$, giving

$$K_X + \Delta \sim R \frac{1}{1 + \lambda} f_*(F' + C_\lambda - B_\lambda) \geq 0.$$  

This is what we wanted to show. $\Box$
The following is an easy property of rational polyhedral cones.

**Lemma 4.3.** Let $\mathcal{C} \subset \mathbb{R}^n$ be a rational polyhedral cone. If $(x_m) \subset \mathcal{C}$ is a sequence converging to $x \in \mathcal{C}$, then there exists an $\varepsilon > 0$ such that for all $m \gg 0$,

$$x_m + \varepsilon(x_m - x) \in \mathcal{C}. $$

The next lemma says that in a certain situation, the pseudo-effective cone has a similar property. In particular, the (closed, convex) set $\{\Upsilon' \in W \mid K_X + A + \Upsilon' \text{ is pseudo-effective}\}$ cannot have a “circular part” like the shaded region in the following picture.

**Lemma 4.4.** Assume Theorems $A_{n-1}$ and $B_{n-1}$.

Let $(X, S + S_1 + \cdots + S_p)$ be a log smooth projective pair of dimension $n$, $A$ an ample $\mathbb{Q}$-divisor on $X$, $W = \langle S, S_1, \ldots, S_p \rangle_{\mathbb{R}}$, and assume $\Upsilon \in \mathcal{L}(W)$, $\Upsilon_m \in W$ are divisors such that

- $\Upsilon_m \to \Upsilon$,
- there is $0 \leq \Sigma \in W$ such that $K_X + A + \Upsilon \sim_{\mathbb{R}} \Sigma$,
- all $K_X + A + \Upsilon_m$ are pseudo-effective.

Assume furthermore the following technical conditions:

- $\text{mult}_S \Upsilon = 1$,
- $\text{mult}_S \Sigma > 0$,
- $\sigma_S(K_X + A + \Upsilon) = 0$.

Then there exists an $\varepsilon > 0$ such that for infinitely many $m$,

$$K_X + A + \Upsilon_m + \varepsilon(\Upsilon_m - \Upsilon)$$

is pseudo-effective.
Sketch of proof. Set \( V = \langle S_1, \ldots, S_p \rangle \subset W \) and \( \Sigma_m = \Sigma + Y_m - Y \). Then \( \Sigma_m \rightarrow \Sigma \) and the \( \Sigma_m \) are pseudo-effective. Let \( Z \in V \) and \( 0 < \varepsilon \ll 1 \) be such that \( Y - \varepsilon Z - S \) is in the interior of \( L(V) \) and \( A' = A + \varepsilon Z \) is still ample. Define

\[
P = \Sigma - (Y - \varepsilon Z - S) + B^S_m(V) \subset W,
\]

and let \( \mathcal{D} = \mathbb{R}_+ \cdot \mathcal{P} \subset W \) be the cone over \( \mathcal{P} \). By Theorem 1.3, \( \mathcal{D} \) is a rational polyhedral cone. And since \( \Sigma - (Y - \varepsilon Z - S) \sim \mathbb{R} k_X + S + A' \), all divisors in \( \mathcal{D} \) are in particular pseudo-effective by the definition of \( B^S_m(V) \).

We claim that setting \( \Gamma_m = \Sigma_m - \sigma_S(\Sigma_m)S \), after passing to a subsequence we have \( \Gamma_m \in \mathcal{D} \) for all \( m \), and \( \Gamma_m \rightarrow \Sigma \). We will not prove the claim here, but let us say how to finish the proof assuming the claim. By Lemma 4.3 there is an \( \varepsilon > 0 \) such that for \( m \gg 0 \), \( \Psi_m = \Gamma_m + \varepsilon(\Gamma_m - \Sigma) \) is pseudo-effective (since it is in \( \mathcal{D} \)). Then

\[
\Sigma'_m := \Sigma_m + \varepsilon(\Sigma_m - \Sigma) = \Psi_m + (1 + \varepsilon)(\Sigma_m - \Gamma_m)
\]
is pseudo-effective too. Hence so is \( K_X + A + \Upsilon_m + \varepsilon(\Upsilon_m - \Upsilon) \sim \mathbb{R} \Sigma_m \), which proves the lemma.

**Theorem 4.5.** Theorem \( A_{n-1} \) and Theorem \( B_{n-1} \) imply Theorem \( B_n \).

**Sketch of proof.** The proof is divided into five steps. Remember that the goal is to show that \( \mathcal{E}_A(V) \) is a rational polytope.

1. Note that by Lemma 4.1 in the definition of \( \mathcal{E}_A(V) \) we may replace linear equivalence by numerical equivalence without changing the set.

2. Show that \( \mathcal{E}_A(V) \) is closed (apply Lemma 4.2).

3. Show that \( \mathcal{E}_A(V) \) is locally a polytope, i.e. extreme points don’t accumulate (apply Lemma 4.4).

4. By compactness, \( \mathcal{E}_A(V) \) is then a polytope.

5. Show that \( \mathcal{E}_A(V) \) is a rational polytope (because the set of numerically trivial divisors in \( V \) is a rational subspace of \( V \)).

The second step is by far the most difficult one.

**5 Proof of Theorem \( A_n \)**

The idea of Lazizi’s proof that Theorem \( A_{n-1} \) and Theorem \( B_{n-1} \) imply Theorem \( A_n \) goes back to Shokurov’s proof of the existence of “pl-flips”. This may roughly be described as follows: Start with a log smooth plt pair \( (X, S + B) \), where \( S \) is a prime divisor, \( [B] = 0 \), and \( S \sim \mathbb{Q} r(K_X + S + B) \) for some rational \( r > 0 \). Then:
• Show that the restricted algebra \( \text{Res}_S R(X, K_X + S + B) \) is finitely generated. This is accomplished by finding a suitable divisor \( C \) on \( S \) such that

\[
\text{Res}_S R(X, K_X + S + B) \cong R(S, K_S + C).
\]

By induction on the dimension, the latter ring is finitely generated, hence so is the former. — One might hope naively that taking \( C = B|_S \) would do. However, it doesn’t. This is where lifting lemmas come in.

• Conclude finite generation of \( R(X, K_X + S + B) \) from that of \( \text{Res}_S R(X, K_X + S + B) \). This step is actually much easier: By the assumption \( S \sim \mathbb{Q} r(K_X + S + B) \), we only need to conclude finite generation of \( R(X, S) \) from that of \( \text{Res}_S R(X, S) \). But it is easy to see that if \( \sigma_1, \ldots, \sigma_\ell \in R(X, S) \) are sections such that \( \sigma_1|_S, \ldots, \sigma_\ell|_S \) generate \( \text{Res}_S R(X, S) \), and \( \sigma \in H^0(X, \mathcal{O}_X(S)) \) is a section whose zero divisor is exactly \( S \), then the set \( \{ \sigma, \sigma_1, \ldots, \sigma_\ell \} \) generates \( R(X, S) \).

Comparing Lazíc’s proof to the special case done by Shokurov, the first step is very similar: apply a lifting lemma and do induction on the dimension. However, in the second step we cannot use an assumption like \( S \sim \mathbb{Q} r(K_X + S + B) \), because this held only by a “relative Picard number one” argument, but in general the Picard number may be arbitrarily large. So \( H^0(X, \mathcal{O}_X(S)) \not\subset R(X, K_X + S + B) \) and we cannot take an extra generator \( \sigma \) like Shokurov did. In order to remedy this situation, we should consider a bigger ring, like \( R(X; K_X + S + B) \). Note that this ring is graded over \( \mathbb{N}^2 \) instead of \( \mathbb{N} \). This was Corti’s original idea: that sometimes higher rank grading is better than rank one.

The first step in the strategy outlined above is carried out by the following lemma.

**Lemma 5.1.** Assume Theorems \( A_{n-1} \) and Theorem \( B_{n-1} \).

Let \( (X, S_1 + \sum_{i=1}^p S_i) \) be a log smooth projective pair of dimension \( n \), where \( S \) and all \( S_i \) are distinct prime divisors. Let \( V = \sum_{i=1}^p \mathbb{R} S_i \subseteq \text{Div}_\mathbb{R}(X) \), let \( A \) be an ample \( \mathbb{Q} \)-divisor on \( X \), let \( B_1, \ldots, B_m \in \mathcal{E}_{S+A}(V) \) be \( \mathbb{Q} \)-divisors, and denote \( D_i = K_X + S + A + B_i \).

Then the ring \( \text{Res}_S R(X; D_1, \ldots, D_m) \) is finitely generated.

A special case of the second step is contained in the next theorem.

**Theorem 5.2.** Assume Theorems \( A_{n-1} \) and Theorem \( B_{n-1} \).

Let \( (X, S_1 + S_2) \) be a log smooth projective pair of dimension \( n \), where \( S_1 \) and \( S_2 \) are distinct prime divisors. Let \( B \) be a \( \mathbb{Q} \)-divisor with \( |B| = 0 \) which is supported on \( S_1 + S_2 \), and let \( A \) be an ample \( \mathbb{Q} \)-divisor. Assume that \( K_X + A + B \sim \mathbb{Q} D \) for some effective \( \mathbb{Q} \)-divisor \( D \) supported on \( S_1 + S_2 \).

Then the ring \( R(X, K_X + A + B) \) is finitely generated.
Obviously, Theorem 5.2 stops short of proving Theorem A in full generality. We indicate how to get rid of the remaining extra assumptions.

- If we have an arbitrary number of components $S_1, \ldots, S_p$ instead of two, the proof goes exactly the same way. It is just more difficult to draw pictures.

- If $K_X + A + B$ is not $\mathbb{Q}$-linearly equivalent to any effective divisor, its section ring is trivial, so we are done anyway. If $K_X + A + B \sim_{\mathbb{Q}} D \geq 0$, we pass to a log resolution of $(X, S_1 + \cdots + S_p + D)$, where we may assume that the support of $D$ is snc.

- If we have an arbitrary number of divisors $B_1, \ldots, B_k$ instead of just one, the proof is again similar to the one above, but a bit more complicated. In particular, the second point of this list (showing that we may assume all $K_X + A + B_i$ to be effective) is not so easy, and we have to apply Theorem B.

Altogether, this shows the following.

**Theorem 5.3.** Theorem $A_{n-1}$ and Theorem $B_n$ imply Theorem $A_n$. 