Mathematical Problem Solving / Summer 2014 / Alex Küronya and Gábor Moussong

Problem Sheet 5

Due date: July 21st

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit. The problems below the line are meant as additional material that might not be discussed in class.

Problems / Mathematical induction and infinite descent

1. Let us draw n pairwise intersecting lines on the plane.
   (1) How many pieces is the plane cut into?
   (2) Show that we can colour the result with two colours in such a way that parts meeting along a line segment will have different colours.
   (3) Can we do the same if we draw lines and circles?

2. Prove the product rule for differentiation: if $f_1, \ldots, f_n \colon \mathbb{R} \to \mathbb{R}$ are differentiable functions, then

$$(f_1 \cdot \cdots \cdot f_n)' = \sum_{i=1}^{n} f_1 \cdot \cdots \cdot f_i' \cdot \cdots \cdot f_n \ .$$

3. The following argument shows that every person in a group has the same eye colour; we will argue by induction. If there is only one person in the group, then there is only one eye colour, and the statement holds. For the induction hypothesis, let us assume that there is only one eye colour within a group of size n; consider now a group of $n+1$ people, we'll number the participants $1, \ldots, n+1$. Look at the sets

$$\{1, 2, \ldots, n\} \quad \text{and} \quad \{2, 3, \ldots, n+1\} \ .$$

Each set contains n people, hence has a unique eye colour, but they overlap, so there must be only one eye colour within the given group of $n+1$ people. Where is the mistake?

4. Verify Newton's binomial theorem

$$(a+b)^n = \sum_{i=1}^{n} \binom{n}{i} a^i b^{n-i}$$

by induction on n.

5. Prove the multinomial theorem

$$(a_1 + \cdots + a_m)^n = \sum_{k_1 + \cdots + k_m = n,\ \text{all } k_i \geq 0} \binom{n}{k_1, \ldots, k_m} a_1^{k_1} \ldots a_m^{k_m} \ ,$$

where

$$\binom{n}{k_1, \ldots, k_m} = \frac{n!}{k_1! \ldots k_m!} \ .$$

Can you give a combinatorial interpretation of the multinomial coefficients?

6. (Bernoulli's inequality) Check that

$$(1+x)^n \geq 1 + nx$$

for all real numbers $x \geq -1$ and all natural numbers n.

7. Show that the sequence

$$c_n = \left(1 + \frac{1}{n}\right)^n$$

is non-decreasing and bounded.

8. Compute the following infinite sums:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \quad , \quad \sum_{n=0}^{\infty} \frac{1}{n^2 + 3n + 2} \quad , \quad \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \; .$$

9. Verify by using infinite descent that the diophantine equation

$$x^3 + 2y^3 + 4z^3 = 0$$

has non non-trivial solutions.

10. Prove that $\sqrt{2}$ is an irrational number.

11. Let d be a positive integer. Show that either $\sqrt{d}$ is again a positive integer, or it is irrational.

12. Show that the diophantine equation

$$x^2 + y^2 = z^2 + v^2$$

has no non-trivial solution.

Extra problems

13. Show that

$$\frac{n^n}{3^n} \leq n! \leq \frac{n^n}{2^n}$$

for all $n \geq 6$.

14. Prove that the diophantine equation

$$x^4 + y^4 = z^4$$

has non non-trivial solution.

15. For a natural number d prove that either d is a perfect cube, or $\sqrt[3]{d}$ is irrational.