Mathematical Problem Solving / Summer 2014 / Alex Küronya and Gábor Moussong

PROBLEM SHEET 4

Due date: July 21st

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit. The problems below the line are meant as additional material that might not be discussed in class.

PROBLEMS / THE PIGEONHOLE AND THE INCLUSION-EXCLUSION PRINCIPLES

1. (Pigeonhole principle) Let n, k be positive integers. We put nk + 1 objects into k boxes. Check that there will be a box containing at least n + 1 objects.

2. Let us draw 9 points in the unit square. Show that there will be three among them which span a triangle of area $\leq \frac{1}{8}$.

3. At a party with $n \ge 2$ guests there will be at least two people who have the same number of acquaintances (acquaintances are assumed to be mutual).

4. In a group of six people there are either three who mutually know each other or there will be three who do not know each other.

5. * Let $\alpha \in \mathbb{R}$ be an irrational number. Show that the fractional parts $\{n\alpha\}$ of its multiples are dense in [0, 1], that is, every subinterval of positive length contains the fractional part of some $n\alpha$.

6. (Ramsey's theorem) Let $m, n \ge 1$ be integers. Show that there is a smallest integer R(m, n) such that in a group of R(m, n) people either there are m who mutually know each other, or there are n who mutually don't. Prove that

$$R(m,n) \le R(m-1,n) + R(m,n-1)$$
.

What is the value of R(m, 1) and R(1, n)?

7. Choose n + 1 numbers from the set $\{1, \ldots, 2n - 1\}$. Prove that there will be two of the chosen ones which sum to 2n.

8. (Inclusion-exclusion principle) Let A_1, \ldots, A_n be finite sets. Prove that

$$\left| \bigcup_{i=1}^{n} A_{i} \right| = \sum_{i=1}^{n} (-1)^{i+1} \sum_{1 \le k_{1} < \dots < k_{i} \le n} |A_{k_{1}} \cap \dots \cap A_{k_{i}}|$$

Start with the cases n = 3 and n = 4.

9. Determine the number of integers between 1 and 1000 that are not divisible by 7,11 or 13.

10. Let A and B be finite sets of size m and n, respectively. Compute the number of surjections of A onto B.

11. How many poker hands are there without aces?

12. Euler's ϕ function is defined by

$$\phi(n) \stackrel{\text{def}}{=} |\{1 \le k \le n \mid k \text{ is coprime to } n\} .$$

whenever n is a positive integer. If $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ is the prime decomposition of n, then show that

$$\phi(n) = n \cdot \prod_{i=1}^k \left(1 - \frac{1}{p_i}\right) \,.$$

EXTRA PROBLEMS

13. Think it through why lossless data compression cannot guarantee actual compression for all inputs.

14. Compute R(4, 2) and R(4, 3).

15. Let $\{x_1, \ldots, x_m\}$ be a finite set. Show that

$$\max\{x_1,\ldots,x_m\} = \sum_{j=1}^n (-1)^{j+1} \sum_{1 \le i_1 < \ldots + i_j \le n} \min\{x_{i_1},\ldots,x_{i_j}\}.$$

16. (Number of derangements) A permutation of the numbers $1, \ldots, n$ is a *derangement*, if no element is left in place. Find the number of derangements on n elements.