Mathematical Problem Solving / Summer 2014 / Alex Küronya and Gábor Moussong

Problem Sheet 3

Due date: July 14th

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit. The problems below the line are meant as additional material that might not be discussed in class.

Problems / Inequalities

1. Prove the inequality between the arithmetic and geometric means:

$$\frac{a+b}{2} \geq \sqrt{ab}$$

for all $a, b > 0$. When do you have equality?

2. (Simplistic isoperimetric inequality) Show that only the square has smallest perimeter among all rectangles of a given area.

3. (Inequality between the arithmetic and geometric means) Let $n \geq 1$ be an integer, $a_1, \ldots, a_n$ positive real numbers. Show that

$$\frac{a_1 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 \ldots a_n}$$

in the following ways.

(1) Use the convexity of the logarithm function.
(2) * Use induction on n.
(3) Use the Lagrange multiplier method applied to the function $f(x_1, \ldots, x_n) = \sqrt[n]{x_1 \ldots x_n}$ and the constraint $g(x_1, \ldots, x_n) = \sum_{i=1}^n x_i - nA$.

4. Let a, b, c be positive real numbers.

(1) Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 1 \ .$$

(2) Find the minimum of the expression

$$\frac{a}{b} + \sqrt[3]{\frac{b}{c}} + \sqrt[4]{\frac{c}{a}} \ .$$

(3) Let $a_1, \ldots, a_n$ be positive real numbers, let $b_1, \ldots, b_n$ be a permutation of the a_i's. Verify that

$$\sum_{i=1}^{n} \frac{a_i}{b_i} \geq n \ .$$

5. Prove the following sequence of inequalities for the positive real numbers a and b.

$$\sqrt{\frac{a^2 + b^2}{2}} \geq \frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2}{\frac{1}{a} + \frac{1}{b}} \ .$$

6. (Cauchy–Schwartz inequality). Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$. Show that

$$\sqrt{a_1^2 + \ldots + a_n^2} \cdot \sqrt{b_1^2 + \ldots + b_n^2} \geq |a_1 b_1 + \ldots + a_n b_n| \ ,$$

and use it to prove the triangle inequality in $\mathbb{R}^n$.

7. State and provee the Cauchy–Schwartz inequality for definite integrals of integrable functions in one variable.

8. Find all continuous functions $f\colon [0,1] \to \mathbb{R}$ such that

$$\int_0^1 f = \frac{1}{2} \quad \text{and} \quad \int_0^1 f^2 = \frac{1}{4} \ .$$

9. Prove that the n-dimensional cube has the minimal sum of lengths of edges connected to ever y vertex among all n-dimensional boxes of the same volume. Is there any other extremal box?

10. Verify the inequalities

(1) $(a+b)(b+c)(c+a) \geq 8abc$,

(2) * $a^2 + b^2 + c^2 \geq ab + bc + ac$

for $a, b, c > 0$.

Extra problems

11. Let $a, b > 0$, let $p \neq 0$ be a real number, and set

$$M_p(a,b) \stackrel{\text{def}}{=} \left(\frac{a^p + b^p}{2} \right)^{\frac{1}{p}} \ .$$

If $p = 0$, then

$$M_0(a,b) \stackrel{\text{def}}{=} \sqrt{ab} \ .$$

Show that $q < p$ implies $M_q(a,b) \leq M_p(a,b)$.

12. ** Find a polynomial $f \in \mathbb{R}[x,y,z]$ which is *not* a positive linear combinations of squares, but is positive for all values of x,y, and z.