Mathematical Problem Solving / Summer 2014 / Alex Küronya and Gábor Moussong

Problem Sheet 2

Due date: July 3rd

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit. The problems below the line are meant as additional material that might not be discussed in class.

Problems / Recursion

1. (Fibonacci numbers) The Fibonacci numbers F_n are defined by the recursive formula

$$F_n \stackrel{\text{def}}{=} F_{n-1} + F_{n-2} \quad \text{for } n \geq 2 \ ,$$

and by setting $F_0 = 0$ and $F_1 = 1$. Verify the following properties of Fibonacci numbers (mathematical induction coupled with their definition is often a good idea).

(1) $\sum_{i=1}^{n} F_i = F_{n+2} - 1$
(2) $\sum_{i=0}^{n-1} F_{2i+1} = F_{2n}$
(3) $\sum_{i=0}^{n} F_{2i} = F_{2n+1} - 1$
(4) $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$
(5) $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{2}}{5}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$
(6) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$
(7) (Cassini's identity) $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n-1}$.
(8) $F_m F_n + F_{m-1}F_{n-1} = F_{m+n-1}$ for all $m, n \geq 1$.

2. Let the sequence x_n be defined by

$$\begin{cases} x_{n+2} \stackrel{\text{def}}{=} 5x_{n+1} + 6x_n & \text{if } n \geq 0 \\ x_0 = -1 \, , \, x_1 = 0 \end{cases} .$$

Compute x_{2014}.

3. * Determine x_{1000} for the sequence given by

$$\begin{cases} x_{n+2} \stackrel{\text{def}}{=} 5x_{n+1} - 6x_n & \text{if } n \geq 0 \\ x_0 = -1 \, , \, x_1 = 0 \end{cases} .$$

4. Find an algorithm for sorting n elements with at most $\frac{n(n-1)}{2}$ comparisons.

5. How many different ways are there to tile a 2-by-10 rectangle with 2×1 dominoes? What is the answer for a 2-by-n rectangle?

6. (Binary search) Let $A[1..n]$ be an array of integers sorted in increasing order.

(1) Find a way to locate a given element x in A if it is there in $\lceil \log_2 n \rceil$ moves, where one move consists of comparing x to an element of the array.

7. (Tower and marbles problem) A company manufacturing marbles tests the strength of its product by dropping a few from various levels of their 100-story office tower, to find the highest floor F from which a marble will not break.

(1) Assuming you have only one marble for testing purposes, how many drops do yo need to find F?
(2) Suppose now that you have two marbles you can use to experiment. How many drops will you need?
(3) How many floors does the tallest building have where we can determine the correct floor F with k drops with one marble?
(4) How many floors does the tallest building have where we can determine the correct floor F with k drops with two marbles? Try to determine the correct number for $k = 2, 3, 4$.
(5) ** Come up with a recursive formula for $n(k, m)$, the greatest number of floors a building can have where we can solve the marble-testing problem with m marbles and k drops.
(6) ** Show that

$$n(k, m) = \sum_{i=0}^{m} \binom{k}{i} - 1 .$$

8. A traveller has a gold chain with 7 rings. He decides to stay at ain inn, where he is to pay one golden ring a day.

(1) What is the least number of rings he has to open in order to be able pay every day?
(2) What is the least number of rings he has to open in order to be able pay every day provided his golden chain consists of n links?
(3) Assuming he can only open 2 rings, what can the maximal number of rings in his chain be if he can pay every day until the chain lasts?
(4) Can you come up with a formula for the case when he can open n rings?

Extra problems

9. (Catalan's identity, general case) $F_n^2 - F_{n-r}F_{n+r} = (-1)^{n-r}F_r^2$ for all $n, r \geq 0$.

10. Consider the recurrence $x_n = 2x_{n-1} - 2x_{n-2}$, $x_0 = 0$, $x_1 = 1$. Show by induction on n that

$$x_n = 2^{n/2} \sin(\frac{n\pi}{4}) .$$

10. (Finding the median) Let $A[1..n]$ be an (unsorted) array filled with not necessarily distinct integers. We say that the element a is the *median* of the elements of A, if

$$|\{1 \leq i \leq n \mid A[i] < a\}| \;<\; \frac{n}{2} \text{ , and}$$
$$|\{1 \leq i \leq n \mid A[i] \leq a\}| \;\geq\; \frac{n}{2} .$$

Find an algorithm that finds the median of A using Cn steps for some positive constant C.