Mathematical Problem Solving / Summer 2014 / Alex Küronya and Gábor Moussong

PROBLEM SHEET 1

Due date: June 30th

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit. The problems below the line are meant as additional material that might not be discussed in class.

Problems

1. One corner of a standard 8-by-8 chessboard are cut out. Can we cover the rest by 2×1 dominoes without overlaps? What happens for 9×9 or $n \times n$ in general?

2. Two opposite corners of a standard 8-by-8 chessboard are cut out. Can we cover the rest by 2×1 dominoes without overlaps? What happens for 9×9 or $n \times n$ in general?

3. In a distant universe, life started out on an $n \times n$ chessboard. At time zero, the whole board was covered with white squares. At time 1, there appeared k black squares out of nowhere; from this point on, life was governed by the following rule: at time m + 1, all squares that had been black at time m remained black, if a white square had at least two black (edge) neighbours at time m, then it turns black.

- (1) Show that it is possible to choose k = n suitable squares at time 1 to be coloured black so that the whole universe turns black after some time.
- (2) Find the minimal number k for which the universe can turn black for a suitable arrangement.

4. (Lunatic asylum problem) Due to some strange events in a remote lunatic asylum, patients and doctors have become indistinguishable just by looking at them; your job will be to find a way to tell them apart. Here is what we know.

- (i) There are altogether 101 people in the institution, there are more doctors than patients. Every person inside the building is either a doctor or a patient.
- (ii) There is one way to extract information from them: you can ask person A what they think about the sanity of person B.
- (iii) Upon being asked, a doctor will always tell the truth, while a patient can say anything.

In the light of this, find solutions to the following questions.

- (1) Find a way to locate a doctor.
- (2) Come up with a procedure to tell apart doctors from patients.
- (3) What can we do if the number of doctors is less than or equal to the number of patients?
- (4) Count how many questions you need to find a doctor, do the same for the separation of doctors from patients.
- (5) Let us assume that we have n people in the asylum; find a way to carry out the same tasks so that the number of questions required is linear in n.

5. (Sorting problems) Let X be a set with a total ordering on it, that is, a relation \leq on pairs of elements of X such that the following properties hold.

- (i) \leq is reflexive, that is, $x \leq x$ for every $x \in X$.
- (ii) \leq is antisymmetric, that is, $x \leq y$ and $y \leq x$ implies x = y.
- (iii) \leq is transitive, that is, for all elements $x, y, z \in X, x \leq y$ and $y \leq z$ imply $x \leq z$.

(iv) For every pair $x, y \in X$, exactly one of the following three options is satisfied: $x \prec y, x = y$, or $y \prec x$.

By a 'sorting problem' we mean that following: we are given n elements of X that we need to put in increasing order. The only operation we can perform is take two elements x and y and compare them with respect to \leq .

- (1) Show that there is a way to sort four elements with five comparisons.
- (2) * How many comparisons do we need to sort five elements?
- (3) How many different ways can we order the numbers $1, 2, \ldots, n$?
- (4) Prove that in order to sort n elements, we need to carry out at least $\log_2 n!$ comparisons.

6. There is a rook on an 8-by-8 chessboard which can only move in the directions south and west. Two players move the rook alternately, in particular, it cannot be left in place. The winner is who moves the rook to the south-west corner.

- (1) Depending on the starting position of the rook, who has a winning strategy?
- (2) How does the answer change if the board is of size 2×2 , 3×3 , or $n \times n$?

7. There are 100 coins on a table. Two players alternate taking coints from the table, at least one, at most 10. The winner is who takes the last coin. Let A be the player who starts, and B the one coming second.

- (1) Who has a winning strategy?
- (2) Analyze the same game if instead of 100 coins, there are 10, 11, or 12 coins on the table.

EXTRA PROBLEMS

8. There are 9 squares in a row. Two players take turns in marking one or two consecutive unmarked squares. The winner is whoever marks the last square.

- (1) Who has a winning strategy?
- (2) How does the answer change if we have 3,4,5,6,7,8, or *n* squares?

9. We place a king on an 8-by-8 chessboard which can only move in the directions south/west/southwest. Two players move the king alternately, in particular, it cannot be left in place. The winner is whoever moves the king to the south-west corner.

- (1) Depending on the starting position of the king, who has a winning strategy?
- (2) How does the answer change as a function of the size of the board?