Introduction to Algebraic Geometry (IAG) / Spring 2014 / Alex Küronya

Homework 3

Due date: March 17$^{\text{th}}$

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. If I, J are ideals in R, then show that $(I\colon J) = (I\colon \sqrt{J})$.

2. Determine the Zariski closure of the following subsets.
   (1) The first quadrant in $\mathbb{A}^2_{\mathbb{R}}$.
   (2) The set of all points on the x axis with rational coordinates in $\mathbb{A}^2_{\mathbb{R}}$.
   (3) The set of all points on the x axis with rational coordinates in $\mathbb{A}^1_{\mathbb{R}}$.
   (4) The set of all points with zero imaginary parts and irrational real parts in $\mathbb{A}^1_{\mathbb{C}}$.
   (5) The set of all points with integral coordinates in $\mathbb{A}^3_{\mathbb{C}}$.
   (6) The set of all points with integral coordinates in $\mathbb{A}^3_{\mathbb{R}}$.
   (7) $\{(a,b) \in \mathbb{A}^2_{\mathbb{R}} \mid a^2+b^2 < 1\}$.

3. Let $I \lhd R$ be an ideal, $\pi\colon R \to R/I$ be the natural ring homomorphism taking $r \mapsto r+I$. Prove that there is a bijective inclusion preserving correspondence between the ideals of R/I and the ideals of R containing I by taking inverse images under π.

4. Let $\phi\colon R \to S$ be a ring homomorphism. Then
   (1) $\ker\phi \stackrel{\text{def}}{=} \{r \in R \mid \phi(r) = 0\}$ is an ideal in R, moreover, all ideals of R are of this form;
   (2) $\operatorname{im}\phi \stackrel{\text{def}}{=} \phi(R)$ is a subring in S, which is not necessarily an ideal.

Show also that ϕ gives rise to an isomorphim (a bijective morphism) of rings

$$\begin{aligned} \overline{\phi}\colon R/\ker\phi &\to \operatorname{im}\phi \\ r+I &\mapsto \phi(r)\ . \end{aligned}$$

5. An element $r \in R$ is called *nilpotent*, if there exists $n \in \mathbb{N}$ for which $r^n = 0$. Verify that the nilpotent elements in a ring form an ideal, which we call the *nilradical* of R, and denote it by $\operatorname{Nil}(R)$. Show also that the ring $R/\operatorname{Nil}(R)$ has no non-zero nilpotent elements.

6. * Prove using Zorn's lemma that

$$\operatorname{Nil}(R) = \bigcap_{P \lhd R \text{ prime}} P\ .$$

7. We define that *Jacobson radical* of R to be the intersection of all maximal ideals of R. Show that for an element $r \in R$, $r \in \operatorname{Rad}(R)$ if and only if $1 - rs \in R^\times$ for all $s \in R$.

8. Show that $\sqrt{I}$ equals the intersection of all prime ideals containing I.

9. Check that the sum of a unit and a nilpotent element is always a unit.

10. * Show that a polynomial $a_0 + a_1x + \ldots + a_nx^n \in R[x]$ is a unit in $R[x]$ if and only if $a_0 \in R$ is a unit, and $a_1, \ldots, a_n \in R$ are nilpotent.

11. ** Prove that an algebraically closed field is infinite.

12. Let (X, τ) be a topological space, $A \subseteq X$ an arbitrary subset. We define the *subspace topology on A with respect to τ* (denoted by (A, τ_A)) via

$$B \in \tau_A \text{ if and only if there exists } U \in \tau \text{ such that } U \cap A = B\ .$$

Show that (A, τ_A) is indeed a topological space, and that τ_A is the smallest topology (with respect to containment) for which the natural inclusion $\iota\colon A \hookrightarrow X$ is continuous.