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REPRESENTATION THEORY/SPRING 2010/ E. Horváth and A. Küronya

PRACTICE SESSION 8

1. (HW) Let L be the real vector space R^3. Define $[x, y] := x \times y$ (cross product of vectors) for $x, y \in L$, and verify that L is a Lie algebra.
Write down the structure constants relative to the usual basis of R^3.
Show that this Lie algebra is isomorphic to $so(3)$, the special orthogonal Lie algebra, consisting of 3×3 antisymmetric matrices of trace zero.

2. Describe all Lie algebras of dimension at most two over the field K.

3. The Lie algebra A_l: special linear Lie algebra $sl(l+1, K)$ consisting of all trace zero matrices over the field K. Show that this is a subalgebra of $gl(l+1, K)$. Determine its dimension, give a basis of it.

4. A bilinear function $f : V_K \times V_K \to K$ is called **skew symmetric (or symplectic)** if $f(u, v) = -f(v, u)$ for all $u, v \in V$.
a) Show that a bilinear function is skew symmetric iff its matrix is skew symmetric. A bilinear function f is called **nondegenerate** if for every nonzero vector $u \in V$ there exists a vector $v \in V$ s.t. $f(u, v) \neq 0$.
b) Show that a bilinear function f is nondegenerate iff its matrix is regular.

5. The symplectic Lie algebra C_l: Let $dim_R V = 2l$. Let $J = \begin{pmatrix} 0_l & I_l \\ -I_l & 0_l \end{pmatrix}$. Let $f(u, v) = u^T J v$ be the antisymmetric, nondegenerate bilinear function with matrix J. Let $Sp(2l, R) = \{G \in GL(n, R) | f(Gu, Gv) = f(u, v)\}$ the group preserving the above symplectic bilinear function.
a)Show that its Lie algebra $sp(2l, R) = \{X \in gl(2l, R) | X^T J = -JX\}$
b)In general $sp(2l, K) = \{X \in gl(2l, K) | X^T J = -JX\}$
c) Show that this algebra consists of matrices $X = \begin{pmatrix} A_l & B_l \\ C_l & D_l \end{pmatrix}$, where $A_l^T = -D_l$, $B_l^T = B_l$, $C_l^T = C_l$.
d) Show that its dimension is $2l^2 + l$ and give a basis of it!

6.(HW) The orthogonal Lie algebra B_l: Let $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0_l & I_l \\ 0 & I_l & 0_l \end{pmatrix}$. Let $o(2l+1, K) = \{X \in gl(2l+1, K) | X^T J + JX = 0\}$. Describe the matrices belonging to this algebra. Show that its dimension is $2l^2 + l$.

7. The orthogonal Lie algebra D_l. $J = \begin{pmatrix} 0_l & I_l \\ I_l & O_l \end{pmatrix}$. Let $o(2l, K) = \{X \in gl(2l, K) | X^T J + JX = 0\}$. Describe the matrices belonging to this algebra. Show that its dimension is $2l^2 - l$.

8.a) Show that $t(n, K)$ the upper triangular matrices in $gl(n, K)$ forms a subalgebra.
b) Show that the stricly upper triangular matrices $n(n, K)$ in $gl(n, K)$ is a subalgebra.
c) Show that the diagonal matrices $d(n, K)$ in $gl(n, K)$ form a subalgebra.

9. Let $H, K \leq L$ subalgebras of the Lie algebra L. Define $[H, K] := \langle [h, k] | h \in H, k \in K \rangle$ be the subspace generated by commutators $[h, k]$.
a) Show that $[d(n, k), n(n, K)] = n(n, K)$.
b) Show that $[t(n, k), n(n, k)] = n(n, k)$.

10. Show that the adjoint map $ad : L \to (Der, [,])$ is a Lie algebra homomorphism.

11. Show that if a matrix $x \in gl(n, F)$ has n distinct eigenvalues $a_1, ..., a_n \in F$ then $adx : gl(n, F) \to gl(n, F)$ has n^2 not necessarily distict eigenvalues $a_i - a_j$, $1 \leq i, j \leq n$.

12. Prove that the set of all inner derivations $adx, x \in L$ is an ideal of $DerL$.

13. a) Prove that $[gl(n, F), gl(n, F)] = sl(n, F)$.
b) Prove that $Z(gl(n, F))$ is just the set of scalar matrices in $gl(n, F)$.
c)Show that if charF is not a divisor of n, then $Z(sl(n, F)) = 0$.
d)Show that if charF divides n then $Z(sl(n, F))$ consists of all scalar matrices in $gl(n, F)$.