Practice Session # 6

Due date: April 13th

The homework problem with an asterisk is the one you are supposed to submit.

Homework

1.* (Campbell–Hausdorff–Baker formula for the Heisenberg group) Let $A, B \in \mathrm{Mat}_n(\mathbb{R})$ be matrices such that $[A[AB]] = [B[AB]] = 0$. Then prove that

$$e^{tA}e^{tB} = e^{tA+tB+\frac{t^2}{2}[AB]}$$

by showing that both sides satisfy the differential equation

$$\frac{dX(t)}{dt} = X(t)(A+B)$$

with the initial condition $X(t) = \mathrm{Id}$.

2. Prove that the exponential map $\exp : \mathfrak{u}(n) \to \mathrm{U}(n)$ is surjective, but not injective.

3. Verify that the exponential map of the Heisenberg group is a bijection.

4. Prove that the following properties characterize the integral curve $\phi_X : \mathbb{R} \to G$ uniquely ($X \in \mathfrak{G}$).
   (1) $\phi'_X(0) = X$,
   (2) $\phi_X : \mathbb{R} \to G$ is a Lie group homomorphism.

5. Let $\Phi : G \to H$ be a Lie group homomorphism, $X \in \mathfrak{g}$. Show that

$$\phi_{(T_1\Phi)(X)} = \Phi \circ \phi_X \ .$$