Practice Session # 5

Due date: March 30$^{\text{th}}$

The homework problem with an asterisk is the one you are supposed to submit.

1. Give an isomorphism between the vector space of local derivations of $\mathcal{O}_{M,x}$ and T_xM. Show that

$$T_xM \simeq \left(\mathfrak{m}_x/\mathfrak{m}_x^2\right)^* \ .$$

2. Consider $C^\infty(M,\mathbb{R})$, the ring of smooth functions on the manifold M, let X be a smooth vector field on M. Check that the assignment $X(f)_m \stackrel{\text{def}}{=} X_m(f)$ gives a derivation of $C^\infty(M,\mathbb{R})$.

3. Prove that the spaces of vector fields on M is isomorphic to $\operatorname{Der}_{\mathbb{R}}(C^\infty(M,\mathbb{R}))$.

4. Let $U \subseteq M$ be a coordinate neighbourhood, $X = \sum_{i=1}^n a_i\partial_i$ and $Y = \sum_{j=1}^n b_j\partial_j$ be vector fields on M. Verify that the vector field corresponding to the derivation $[X,Y] \stackrel{\text{def}}{=} XY - YX$ is

$$\sum_{i,j=1}^n \left(a_j(\partial_j b_i) - b_j(\partial_j a_i)\right)\partial_i \ .$$

Definition. Let G be a Lie group. A vector field X on G is called *left-invariant* if $(T_gL_h)X_h = X_{gh}$ for every $g,h \in G$.

5. Show that the real vector space of left-invariant vector fields on a Lie group G forms a Lie algebra of dimension $\dim G$ with the bracket operation introduced above. Consequently, deduce that for every $X \in T_1G$ there exists a unique left-invariant vector field Y on G for which $(Y)_1 = X$.

Homework

Definition. A matrix $A \in \operatorname{Mat}_n(\mathbb{R})$ is said to be *unipotent*, if $A - \operatorname{Id}$ is nilpotent.

6. Let $A \in \operatorname{Mat}_n(\mathbb{R})$ be a unipotent matrix. Check that $\log A$ is convergent, nilpotent and $\exp(\log A) = A$ holds.

7. Show that if $N \in \operatorname{Mat}_n(\mathbb{R})$ is nilpotent, then e^N is unipotent and $\log(\exp N)) = N$.

8. For every $A \in \operatorname{GL}_n(\mathbb{R})$ there exists $X \in \operatorname{Mat}_n(\mathbb{R})$ such that $e^X = A$.

9. Give an example for a matrix Lie group G and $A \in \operatorname{Mat}_n(\mathbb{R})$ for which $e^A \in G$, but $A \notin \mathfrak{g}$ (that is, there exists $t \in \mathbb{R}$ such that $e^{tA} \notin G$).

10. Check that the Pauli matrices

$$E_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} , \ E_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \ E_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} ,$$

form a basis of $\mathfrak{s}u(2)$. Compute $[E_i,E_j]$ for all pairs. Prove that there exists a linear isomorphism $\phi : \mathfrak{s}u(2) \longrightarrow \mathbb{R}^3$ such that $\phi([A,B]) = \phi(A) \times \phi(B)$ (where $\times$ denotes the usual cross product of vectors).

11. Verify that $\mathfrak{s}u(2) \simeq \mathfrak{s}o(3)$.

12. * Prove that for all matrices $A,B \in \operatorname{Mat}_n(\mathbb{R})$, one has

$$(\operatorname{ad}_A)^m(B) = \sum_{k=0}^m \binom{m}{k} A^k B(-A)^{m-k} \ .$$

Use the above identity to conclude that $e^{\operatorname{ad}_A}(B) = \operatorname{Ad}_{e^A}(B) = e^ABe^{-A}$.

13. Let G be a Lie group, $\rho_i : G \to \mathrm{GL}(V_i)$ $(i = 1, 2)$ two representations of G. Recall that the tensor product of the two representations is given by

$$\begin{array}{rcl} \rho_1 \otimes \rho_2 : G & \longrightarrow & GL(V_1 \otimes V_2) \\ g & \mapsto & \rho_1(g) \otimes \rho_2(g) \; . \end{array}$$

Let $\mathfrak{g}$ be the Lie algebra of G, then the ρ_i's induce Lie algebra representations $\tilde{\rho}_i : \mathfrak{g} \to \mathfrak{g}l(V_i)$ via the tangent functor. Describe the induced action of $\mathfrak{g}$ on $V_1 \otimes V_2$.

14. Let $\exp : \mathfrak{s}l(2, \mathbb{R}) \to \mathrm{SL}(2, \mathbb{R})$ denote the exponential map.

(1) If $A \in \mathfrak{s}l(2, \mathbb{R})$ then prove that the eigenvalues of A are either of the form $\{\lambda, -\lambda\}$ or $\{i\lambda, -i\lambda\}$ where $\lambda \in \mathbb{R}$.

(2) Describe all possible sets of eigenvalues of matrices in $\mathrm{SL}(2, \mathbb{R})$.

(3) Compute the image of exp and conclude that it is not dense in $\mathrm{SL}(2, \mathbb{R})$.

15. (i) Let $A \in \mathrm{SO}(3)$. Prove that A has an eigenvalue with absolute value 1.
(ii) Find an orthonormal basis with respect to which A has the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$$

(iii) Prove that $\exp : \mathfrak{s}o(3) \to \mathrm{SO}(3)$ is surjective.