Practice Session # 4

Due date: March 23$^{\mathbf{th}}$

The homework problem with an asterisk is the one you are supposed to submit.

1. (Matrix exponentials) Let $A \in \mathrm{Mat}_n(\mathbb{R})$ (or $\mathrm{Mat}_n(\mathbb{C})$). We define

$$e^A \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{A^k}{k!} .$$

(1) Show that the above sum converges for every matrix A, and the limit is a continuous function of A.
(2) $(e^A)^* = e^{(A^*)}$.
(3) e^A is invertible, $(e^A)^{-1} = e^{-A}$.
(4) For every $u, t \in \mathbb{R}$ (or $\mathbb{C}$), we have that $e^{(u+t)A} = e^{uA}e^{tA}$.
(5) If $AB = BA$, then $e^{A+B} = e^A e^B = e^B e^A$.
(6) For every $C \in \mathrm{GL}(n)$, one has $e^{CAC^{-1}} = Ce^A C^{-1}$.
(7) $\|e^A\| \leq e^{\|A\|}$.
(8) The map $t \mapsto e^{tA}$ gives rise to a smooth curve, $\frac{d}{dt}e^{tA} = e^{tA}A = Ae^{tA}$.

2. Let $A \in \mathrm{Mat}_n(\mathbb{C})$. Prove that there exists a unique pair of matrices $S, N \in \mathrm{Mat}_n(\mathbb{C})$ such that $A = S+N$, $SN = NS$, S is diagonalizable, and N is nilpotent.

3. (Computing matrix exponentials) Determine e^A if A is a diagonalizable/nilpotent matrix. Sue the previous exercise if needed to compute e^A for the following matrices:

$$\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} , \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} , \begin{pmatrix} d & e \\ 0 & d \end{pmatrix} .$$

4. Accepting Problems 11. and 12., compute the Lie algebras of the matrix Lie groups $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$.

5. Let M be a smooth manifold, $x \in M$.
(1) Verify that the set of germs of C^∞ functions $\mathcal{O}_{M,x}$ at x is a ring with the obvious operations.
(2) Check that $\mathrm{ev}_x : \mathcal{O}_{M,x} \to \mathbb{R}$, the evaluation map at x is an $\mathbb{R}$-algebra homomorphism, and that its kernel $\mathfrak{m}_x$ is the unique maximal ideal of $\mathcal{O}_{M,x}$, thus observing that $\mathcal{O}_{M,x}$ is a local ring.

6. Let M be an n-dimensional smooth manifold, $x \in M$, $x_1, \dots x_n$ local coordinates near x. Prove that
(1) The germs of $x_1, \dots, x_n$ generate $\mathfrak{m}_x$.
(2) $\dim_{\mathbb{R}} \mathfrak{m}_x/\mathfrak{m}_x^2 = n$, and the germs of the x_i's form a basis of $\mathfrak{m}_x/\mathfrak{m}_x^2$.

Definition. Let k be an arbitrary field, A a k-algebra. A *derivation of* A is a k-linear map $D : A \to A$ satisfying the Leibniz rule, that is, if $f, g \in A$ then

$$D(fg) = D(f)g + fD(g) .$$

7. Check that $D(\alpha) = 0$ for all $\alpha \in k$. Show that if D_1, D_2 are derivations of A, then so is $D_1 \circ D_2 - D_2 \circ D_1$. If $A = k[x_1, \dots x_n]$ a polynomial ring then the map

$$f \mapsto \sum_{i=1}^{n} g_i \frac{\partial f}{\partial x_1} \quad (g_i \in k[x_1, \dots, x_n])$$

is a derivation of A. Find an example of derivations D_1, D_2 of a polinomial ring for which $D_1 \circ D_2$ is *not* a derivation.

8. Let $z \in \mathbb{C}$. Check that the power series

$$\log z \stackrel{\text{def}}{=} \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(z-1)^m}{m}$$

converges absolutely to an analytic function on the disc $|z-1| < 1$. On this domain $e^{\log z} = z$. If $|w| < \log 2$, then $|e^w - 1| < 1$ and $\log e^w = w$.

9. (Matrix logarithm) Let $A \in \mathrm{Mat}_n(\mathbb{C})$. Show that the series

$$\log A \stackrel{\text{def}}{=} \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A - \mathrm{Id})^m}{m}$$

converges to a continuous function whenever $\|A - \mathrm{Id}\,\| < 1$. Furthermore

(1) If $\|A - \mathrm{Id}\,\| < 1$ then $e^{\log A} = A$.
(2) If $\|A\| < \log 2$ then $\|e^A - \mathrm{Id}\,\| < 1$ and $\log e^A = A$.
(3) There exists a real constant C_A such that

$$\|\log(\mathrm{Id} + A) - A\| \leq C_A \cdot \|A\|^2$$

provided $\|A\| \leq \frac{1}{2}$.

10. (Lie product formula) For all pairs of (not necessarily commuting) matrices $X, Y \in \mathrm{Mat}_n(\mathbb{C})$,

$$e^{X+Y} = \lim_{m\to\infty} \left(e^{\frac{X}{m}} e^{\frac{Y}{m}}\right)^m .$$

11. Let $G \leqslant \mathrm{GL}(n, \mathbb{R})$ be a closed Lie subgroup, $X \in \mathrm{Mat}_n(\mathbb{R})$. Then the curve $t \mapsto e^{tX}$ is tangent to the manifold G (equipped with its submanifold structure from $\mathrm{GL}(n, \mathbb{R})$) at $t = 0$ if and only if $e^{tX} \in G$ for all $t \in G$.

12. * Prove that $\det e^A = e^{\mathrm{Tr}\, A}$ for every $A \in \mathrm{GL}(n, \mathbb{C})$.

13. Let $U \subseteq \mathbb{R}^n$ be an open neighbourhood of the origin, $f \in C^\infty(U)$, and assume that $f(0, x_2, \ldots, x_n) = 0$ whenever $(0, x_2, \ldots, x_n) \in U$. Show that

$$\tilde{f}(x_1, \ldots, x_n) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{x_1} f(x_1, \ldots, x_n) & \text{if } x_1 \neq 0 \\ \frac{\partial f}{\partial x_1}(0, x_2, \ldots, x_n) & \text{if } x_1 = 0 \end{cases}$$

defines a smooth function on U.

14. Determine the Lie algebra of the Heisenberg group.