PRACTICE SESSION # 3

Due date: March 16th

The homework problem with an asterisk is the one you are supposed to submit.

1. Let G be a topological group. Then the following hold.
   (1) Every open subgroup of G is closed, conversely, every closed subgroup of finite index is open.
   (2) For a subgroup $H \leqslant G$, H is open if and only if G/H is discrete.
   (3) All subgroups containing a neighbourhood of 1_G are open.

DEFINITION. A topological space X is called *totally disconnected*, if every connected component of X has one point.

2. Show that the connected component G° of a topological group G is a closed connected normal subgroup of G for which G/G° is a totally disconnected Hausdorff group. In addition, show that the components of G are exactly the cosets of G°, and every open subgroup of G contains G°.

3. It is known that $E \stackrel{\text{def}}{=} \overline{\{1_G\}}$ is a normal subgroup of G. Verify that G/E is the *universal Hausdorff subgroup on* G in the following sense. For every morphism of topological groups $\phi : G \to H$ with H Hausdorff, there exists a unique morphism of topological groups $\widetilde{\phi} : G/E \to H$ such that $\widetilde{\phi} \circ q = \phi$.

4. In this exercise we construct a very important map of topological groups $\phi : \mathrm{SU}(n, \mathbb{C}) \to \mathrm{SO}(3, \mathbb{R})$, which is a twofold covering space.
   (1) Let V be the set of 2×2 self-adjoint complex matrices A with $\mathrm{Tr}(A) = 0$. Define a scalar product on V by setting
   $$\langle A, B \rangle \stackrel{\text{def}}{=} \frac{1}{2} \mathrm{Tr}(AB) \ .$$
   Prove that the so-called Pauli matrices
   $$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ A_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \ , \ A_1 = \begin{pmatrix} 1 & 0 \\ & -1 \end{pmatrix}$$
   for an orthonormal basis of V. We will identify V with $\mathbb{R}^3$ via this basis.
   (2) Show that the function
   $$\begin{aligned} \phi : \mathrm{SU}(2) & \longrightarrow \mathrm{End}_{\mathbb{R}}(V) \\ U & \mapsto (A \mapsto UAU^{-1}) \end{aligned}$$
   is a well-defined morphism of topological groups with $\mathrm{im}\, \phi \subseteq \mathrm{SO}(3, \mathbb{R})$, that is, it gives rise to a morphism of topological groups $\phi : \mathrm{SU}(2, \mathbb{C}) \to \mathrm{SO}(3, \mathbb{R})$ (you can make use of the fact that $\mathrm{SU}(2, \mathbb{C})$ is connected).
   (3) Prove that ϕ is two-to-one and onto (if you know what a covering space is, show that ϕ is a covering space as well).
   (4) For $w, z \in \mathbb{C}$ arbitrary, with $|w|^2 + |z|^2 = 1$, check that the matrix
   $$\begin{pmatrix} w & -\overline{z} \\ z & \overline{w} \end{pmatrix}$$
   is an element of $\mathrm{SU}(2, \mathbb{C})$. Conversely, verify that every matrix $U \in \mathrm{SU}(2, \mathbb{C})$ can be expressed in the above form for a unique pair of complex numbers (w, z). (This way one can think of $\mathrm{SU}(2, \mathbb{C})$ as $\mathbb{S}^3 \subseteq \mathbb{C}^2 = \mathbb{R}^4$.)

5. (*i*) Let $n \geq 2$, $v \in \mathbb{R}^n$ an arbitrary vector. Check that there exists a continuous path $\gamma : I \to \mathrm{SO}(n, \mathbb{R})$ with $\gamma(0) = \mathrm{Id}$ and $\gamma(1)v = (1, 0, \ldots, 0)$.
(*ii*) Using induction on n, show that $\mathrm{SO}(n, \mathbb{R})$ is a connected topological group.
(*iii*) Prove that $\mathrm{O}(n, R)$ has exactly two connected components, one of which is $\mathrm{SO}(n, \mathbb{R})$.

6. Which of the groups $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{R})$, $\mathbb{R}^n$, $\mathbb{S}^1$, $\mathrm{O}(n, \mathbb{R})$, $\mathrm{SO}(n, \mathbb{R})$, $\mathrm{U}(n, \mathbb{C})$, $\mathrm{SU}(n, \mathbb{C})$, $\mathrm{Sp}(n, \mathbb{R})$ are compact?

7. * (Polar decomposition and connectedness of $\mathrm{SL}(n,\mathbb{R})$). Using the following line of thought, prove that $\mathrm{SL}(n,\mathbb{R})$ is a connected topological group.

(1) For a symmetric positive definite matrix $A \in \mathrm{GL}(n,\mathbb{R})$ there exists an orthogonal matrix B such that $A = BDB^{-1}$, with D a diagonal matrix with strictly positive diagonal entries $\lambda_1, \ldots, \lambda_n$.
(2) With A as above, there exists a *unique* matrix $A^{1/2} \in \mathrm{GL}(n,\mathbb{R})$ for which $(A^{1/2})^2 = A$.
(3) For a matrix $M \in \mathrm{GL}(n,\mathbb{R})$ there exists a unique pair of matrices (B, A) where $B \in \mathrm{SO}(n,\mathbb{R})$, A is symmetric and positive definite, and $M = BA$. In addition, $\det A = 1$.
(4) Using the fact that $\mathrm{SO}(n,\mathbb{R})$ is connected, prove that $\mathrm{SL}(n,\mathbb{R})$ is a connected topological group.

8. Prove that every discrete topological space is totally disconnected. Is the converse true?

9. Let X, $\{X_\alpha \mid \alpha \in I\}$ be (non-empty) totally disconnected topological spaces.

(1) Show that every subspace of X is totally disconnected.
(2) Prove that $\prod_{\alpha \in I} X_\alpha$ is totally disconnected if and only if every X_α is so.

10. Let G be a connected topological group, $N \lhd G$ a discrete normal subgroup. Show that $N \subseteq Z(G)$.

11. Prove that the topological groups $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{U}(n,\mathbb{C})$, $\mathrm{SU}(n,\mathbb{C})$ are all connected.

12. Show that $\mathrm{GL}(n,\mathbb{R})$ is not connected, but has the two connected components

$$\mathrm{GL}(n,\mathbb{R})^+ = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid \det A > 0\} \text{ and } \mathrm{GL}(n,\mathbb{R})^- = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid \det A < 0\} .$$

13. (The group of rotations in the plane) Prove that the matrix

$$M(\phi) \stackrel{\text{def}}{=} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

belongs to $\mathrm{SO}(2,\mathbb{R})$. Check that $M(\phi_1)M(\phi_2) = M(\phi_1 + \phi_2)$. Next, verify that every element of $\mathrm{O}(2,\mathbb{R})$ is of one of the forms

$$M^+(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \text{ or } M^-(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} .$$

14. Is the Heisenberg group connected? Is it compact?