Practice Session # 2

Due date: March 2nd

The homework problem with an asterisk is the one you are supposed to submit.

1. Let G be a topological group, $H \leqslant G$. Prove that $\overline{H} \subseteq G$ is again a subgroup. If H was a normal subgroup in G, then so is $\overline{H}$.

2. Show that a surjective map, which is open/closed, is a quotient map.

3. Prove that the product of two open maps is open as well.

4. Let (G, τ) be a topological group with multiplication μ and inversion i. Let $H \leqslant G$. Check that $(H, \tau|_H)$ with the operations $\mu|_H$ and $i|_H$ is again a topological group.

5. (Orthogonal and special orthogonal groups) Let $A \in \mathrm{Mat}_n(\mathbb{R})$, with column vectors $a_1, \ldots, a_n$. We denote the usual scalar product of the vectors $x, y \in \mathbb{R}^n$ by $\langle x, y \rangle$.
   (1) Show that the following conditions on A are equivalent.
      (a) The vectors $a_1, \ldots, a_n$ are orthonormal, i.e. $\langle a_i, a_j \rangle = \delta_{ij}$ for every $1 \leq i, j \leq n$.
      (b) We have $\langle Ax, Ay \rangle = \langle x, y \rangle$ for every $x, y \in \mathbb{R}^n$.
      (c) $A^T A = \mathrm{Id}$.

   A matrix satisfying these conditions is called *orthogonal*, the set of $n \times n$ orthogonal matrices is denoted by $O(n, \mathbb{R})$. An interesting subset of $O(n, \mathbb{R})$ is the collection of *special orthogonal* matrices,
$$SO(n, \mathbb{R}) \stackrel{\text{def}}{=} \{A \in O(n, \mathbb{R}) \mid \det A = 1\} .$$
   (2) Prove that the determinant of an orthogonal matrix equals ± 1.
   (3) Show that $O(n), SO(n) \leqslant \mathrm{GL}(n, \mathbb{R})$, in particular, they are topological groups themselves.

6. (Unitary and special unitary groups) Let $A \in \mathrm{Mat}_n(\mathbb{C})$, with column vectors $a_1, \ldots, a_n$. We denote the usual Hermitian scalar product of the vectors $w, z \in \mathbb{C}^n$ by $\langle w, z \rangle \stackrel{\text{def}}{=} \sum_{i=1}^n w_i \overline{z}_i$.
   (1) Show that the following conditions on A are equivalent.
      (a) The vectors $a_1, \ldots, a_n$ are orthonormal, i.e. $\langle a_i, a_j \rangle = \delta_{ij}$ for every $1 \leq i, j \leq n$.
      (b) We have $\langle Aw, Az \rangle = \langle w, z \rangle$ for every $w, z \in \mathbb{C}^n$.
      (c) $A^* A = \mathrm{Id}$.

   Here $A^* \stackrel{\text{def}}{=} \overline{A}^T$ denotes the adjoint of A. A matrix satisfying these conditions is called *unitary*, the set of $n \times n$ unitary matrices is denoted by $U(n, \mathbb{C})$. A very important subset of $U(n, \mathbb{C})$ is the collection of *special unitary* matrices,
$$SU(n, \mathbb{C}) \stackrel{\text{def}}{=} \{A \in U(n, \mathbb{C}) \mid \det A = 1\} .$$
   (2) Prove that $|\det A| = 1$ whenever $A \in U(n, \mathbb{C})$.
   (3) Show that $U(n), SU(n) \leqslant \mathrm{GL}(n, \mathbb{C})$, in particular, they are topological groups themselves.

7. (Symplectic groups) Consider the skew-symmetric bilinear form S on $\mathbb{R}^{2n}$ given by
$$S(x, y) \stackrel{\text{def}}{=} \sum_{k=1}^{n} (x_k y_{n+k} - x_{n+k} y_k) .$$
   (1) If $J \in \mathrm{Mat}_{2n}(\mathbb{R})$ is the matrix
$$J = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} ,$$
   then show that $S(x, y) = \langle x, Jy \rangle$ for all $x, y \in \mathbb{R}^{2n}$.

(2) Check that a matrix $A \in \mathrm{Mat}_{2n}(\mathbb{R})$ preserves S (in the sense that $S(Ax, Ay) = S(x, y)$ for all vectors $x, y \in \mathbb{R}^{2n}$) precisely if $A^T J A = J$.

The group of $2n \times 2n$ matrices satisfying the above equivalent conditions is called the *symplectic group*, and is denoted by $\mathrm{Sp}(n, \mathbb{R})$.

Homework

8. Let $H \lhd G$ be a normal subgroup of a topological group, $\phi : G \to L$ a morphism of topological groups such that $H \subseteq \ker \phi$. Show that there exists a unique morphism $\overline{\phi} : G/H \to L$ for which $\overline{\phi} \circ q = \phi$.

9. Let X be a topological space. Prove that the following are equivalent.
   (1) X is Hausdorff.
   (2) The diagonal $\Delta(X) \stackrel{\text{def}}{=} \{(x, x) \mid x \in X\} \subseteq X \times X$ is a closed subset.
   (3) For any topological space Y and any two maps $f, g : Y \to X$, the subset $\{y \in Y \mid f(y) = g(y)\}$ is closed in Y.

10. (Heisenberg group) The collection $H \subseteq \mathrm{GL}_3(\mathbb{R})$ of 3×3 real matrices of the form

$$A \stackrel{\text{def}}{=} \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

is a subgroup of the general linear group. Give an explicit formula for A^{-1}.

11. * Let $\phi : G \to H$ be a morphism of topological groups, $j : \Im\phi \to H$ the inclusion, $q : G \to G/\ker \phi$ the quotient map. Show that there exists a unique morphism of topological groups $\widetilde{\phi}$ such that

$$j \circ \widetilde{\phi} \circ q = \phi \ .$$

The morphism $\widetilde{\phi}$ is always bijective (but not necessarily a homeomorphism). If ϕ is open/closed then $\widetilde{\phi}$ is a homeomorphism.

12. Show that $\mathrm{GL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{R}) \simeq \mathbb{R}^\times$ as topological groups via the determinant map.

13. Let $X \stackrel{\text{def}}{=} \mathbb{Z}$, with the usual addition of integers μ and additive inverse i, τ the topology generated by the collection $\{[n, +\infty) \mid n \in \mathbb{Z}\}$. Show that μ is continuous with respect to τ, but i is *not*.

14. Let G be a topological group, $\mathcal{N}$ an open neighbourhood basis of $1_G \in G$.
   (1) Show that for every $U \in \mathcal{N}$ there exist $V, W \in \mathcal{N}$ such that $V^{-1} \subseteq U$ and $W^2 \subseteq U$.
   (2) If U, V are open neighbourhoods of 1, then so are $U \cap V$ and U^{-1}.
   (3) Every open neighbourhood U of 1 contains a symmetric open neighbourhood (that is, one for which $W^{-1} = W$).