PRACTICE SESSION # 1

Due date: February 23rd

The homework problem with an asterisk is the one you are supposed to submit.

1. Let X be a set, G an arbitrary group. Check that the evaluation action $\mathrm{ev} : \mathrm{Aut}(X) \times X \to X$ and the trivial action $\mathrm{triv} : G \times X \to X$ are indeed group actions.

2. Consider the complex upper half-plane $\mathfrak{H} \stackrel{\text{def}}{=} \{z = x + iy \,|\, y > 0\}$ with the function

$$\begin{aligned} \alpha : \mathrm{SL}_2(\mathbb{R}) \times \mathfrak{H} &\longrightarrow \mathfrak{H} \\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) &\mapsto \frac{az+b}{cz+d} . \end{aligned}$$

Check that α is indeed a group action.

3. Let G be a group, k be an arbitrary field, V a k-vector space (not necessarily finite-dimensional). Show that every representation $\rho : G \to \mathrm{GL}_k(V)$ gives rise to a unique k-algebra homomorphism $\widetilde{\rho} : k[G] \to \mathrm{End}_k(V)$ with $\tilde{\rho}|_G = \rho$, and vice versa ($k[G]$ denotes the appropriate group algebra).

4. For given representations $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$, we define a natural representation of G on $\mathrm{Hom}_k(V, W)$. If $g \in G$, $\phi \in \mathrm{Hom}_k(V, W)$, and $v \in V$, then $\tau : G \to \mathrm{GL}(\mathrm{Hom}_k(V, W))$ is given by

$$(\tau(g)(\phi))(v) \stackrel{\text{def}}{=} (\sigma(g))(\phi(\rho(g^{-1})(v))) .$$

Check that τ is indeed a representation of G. Compute the special case when σ is the trivial representation on k. The result is called the dual representation of ρ.

5. Continuing the previous exercise, let $\tau : G \to \mathrm{GL}(V^*)$ the the representation dual to ρ. Prove that for every $v \in V$, $w^* \in V^*$, and $g \in G$,

$$\langle (\tau(g))(w^*), (\rho(g))(v) \rangle = \langle w^*, v \rangle .$$

6. Let V be a finite-dimensional vector space over k, $\phi \in \mathrm{Hom}_k(V, V)$ a projection, that is, a linear map with $\phi^2 = \phi$. Prove that

$$\dim \phi(V) = \mathrm{Tr}\, \phi .$$

DEFINITION. Let $\rho : G \to \mathrm{GL}_k(V)$ be a representation of G on a finite-dimensional vector space V. Then $\chi_\rho : G \to k^\times$, the *character of* G is defined by

$$\chi_\rho(g) \stackrel{\text{def}}{=} \mathrm{Tr}\, \rho(g) .$$

7. Let $\rho : G \to \mathrm{GL}(V)$ be a finite-dimensional representation of the finite group G, let

$$V^G \stackrel{\text{def}}{=} \{v \in V \,|\, \rho(g)(v) = v \text{ for all } g \in G\}$$

be the *invariant subspace* of ρ. Using the averaging map $\phi : V \to V$ given by

$$\phi(v) \stackrel{\text{def}}{=} \frac{1}{|G|} \sum_{g \in G} \rho(g)(v)$$

show that

$$\dim V^G = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) .$$

HOMEWORK

DEFINITION. Let G be a group, $S \subseteq G$ a subset (not necessarily a subgroup). The *centralizer of* S is defined as

$$Z(S) \stackrel{\text{def}}{=} \{g \in G \,|\, gs = sg \text{ for all } s \in S\} ,$$

while the *normalizer of* S is

$$N(S) \stackrel{\text{def}}{=} \{g \in G \mid gS = Sg\} \ .$$

The centralizer $Z(G)$ of G is called the *center of* G.

8. (i) Prove that $Z(G) \lhd G$.
(ii) Show that for every $S \subseteq G$, one has $Z(S) \leqslant G$ and $N(S) \leqslant G$.
(iii) Let $S \leqslant G$. Prove that $N(S)$ is the largest subgroup of G, in which S is a normal subgroup.

9. (Class formula) Let G be a finite group with center Z, $x_1, \dots, x_m$ a complete system of representatives of the orbits of $G \setminus Z$. Then

$$|G| \ = \ |Z| + \sum_{i=1}^{m} |G : Z(x_i)| \ .$$

10. Let G be a finite group, $H \leqslant G$. Consider the action of H on G by left translation. What does the orbit formula say in this case?

11.* Let G be an arbitrary group, and $\mathcal{S}$ the set of all subgroups of G. Show that

$$\begin{array}{rcl} \alpha : G \times \mathcal{S} & \longrightarrow & \mathcal{S} \\ (g, H) & \mapsto & gHg^{-1} \end{array}$$

defines an action of G on the collection $\mathcal{S}$. When does the orbit of a subgroup G consist of one element?

Let us now assume that G is finite, and $|G| = p^k$ for some prime number p. Prove that the number

$$\text{number of subgroups of } G \ - \ \text{number of normal subgroups of } G$$

is divisible by p.

12. Let G be a finite group with p^k elements, p a prime. Show that $p \big| |Z(G)|$, in particular, $Z(G) \neq 1$.

13. Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ be two representations. Verify that $\tau : G \to \mathrm{GL}(V \otimes W)$ defined as

$$\tau(g) \stackrel{\text{def}}{=} \rho(g) \otimes \sigma(g)$$

is a again a representation of G.

14. Let X be a finite set, $\alpha : G \times X \to X$ a group action giving rise to a homomorphism $\tilde{\alpha} : G \to \mathrm{Aut}(X)$. Using $\tilde{\alpha}$, we can define a linear representation of G, which is called the associated *permutation representation* as follows. Let V be the vector space over k with basis $\{e_x \mid x \in X\}$. Then $\pi : G \to \mathrm{GL}(V)$ is defined by

$$\pi(g)(\sum_{x \in X} a_x e_x) \stackrel{\text{def}}{=} \sum_{x \in X} a_x e_{\alpha(g)(x)} \ .$$

Check that π is indeed a representation.

Alternatively, we could set V' to be the vector space of functions $\phi : X \to k$, and define $\pi' : G \to \mathrm{GL}(V')$ as

$$(\pi'(g)(\phi))(x) \stackrel{\text{def}}{=} \phi((\tilde{\alpha}(g^{-1}))(x)) \ .$$

Verify that π' is also a representation and show that the representations π and π' are isomorphic by identifying e_x with the characteristic function of x.

15. Let $\alpha : G \times X \to X$ be an action of the finite group G on the finite set X. Prove that the number of orbits of α equals

$$\frac{1}{|G|} \sum_{g \in G} |\,\mathrm{Fix}(g)| \ .$$