INTRODUCTION TO TOPOLOGY

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1. Basic concepts

Topology is the area of mathematics which investigates continuity and related concepts. Important fundamental notions soon to come are for example open and closed sets, continuity, homeomorphism.

Originally coming from questions in analysis and differential geometry, by now topology permeates mostly every field of math including algebra, combinatorics, logic, and plays a fundamental role in algebraic/arithmetic geometry as we know it today.

Definition 1.1. A *topological space* is an ordered pair (X, τ), where X is a set, τ a collection of subsets of X satisfying the following properties

(1) $\emptyset, X \in \tau$,
(2) $U, V \in \tau$ implies $U \cap V$,
(3) $\{U_\alpha \,|\, \alpha \in I\}$ implies $\cup_{\alpha \in I} U_\alpha \in \tau$.

The collection τ is called a *topology* on X, the pair (X, τ) a *topological space*. The elements of τ are called *open sets.*

A subset $F \subseteq X$ is called *closed*, if its complement $X - F$ is open.

Although the official notation for a topological space includes the topology τ, this is often suppressed when the topology is clear from the context.

Remark 1.2. A quick induction shows that any finite intersection $U_1 \cap \cdots \cap U_k$ of open sets is open. It is important to point out that it is in general not true that an arbitrary (infinite) union of open sets would be open, and it is often difficult to decide whether it is so.

Remark 1.3. Being open and closed are not mutually exclusive. In fact, subsets that are both open and closed often exist, and play a special role.

The collection of closed subsets in a topological space determines the topology uniquely, just as the totality of open sets does. Hence, to give a topology on a set, it is enough to provide a collection of subsets satisfying the properties in the exercise below.

Exercise 1.4. *Prove the following basic properties of closed sets. If (X, τ) is a topological space, then*

(1) $\emptyset, X$ are closed sets,
(2) if $F, G \subseteq X$ are closed, then so is $F \cup G$,
(3) if $\{F_\alpha \mid \alpha \in I\}$ is a collection of closed subsets, then $\cap_{\alpha \in I} F_\alpha$ is closed as well.

Example 1.5 (Discrete topological space)**.** Let X be an arbitrary set, $\tau \stackrel{\text{def}}{=} 2^X$, that is, we declare every subset of X to be open. One checks quickly that (X, τ) is indeed a topological space.

Example 1.6 (Trivial topology)**.** Considering the other extreme, the pair $(X, \{\emptyset, X\})$ is also a topological space, with $\emptyset$ and X being the only open subsets.

The previous two examples are easy to understand, however not that important in practice. The primordial example of a very important topological space coming from analysis is the real line. In fact $\mathbb{R}^1$ and its higher-dimensional analogues are the prime source of our topological intuition. However, since there are copious examples of important topological spaces very much unlike $\mathbb{R}^1$, we should keep in mind that not all topological spaces look like subsets of Euclidean space.

Example 1.7. Let $X = \mathbb{R}^1$. We will define a topology on $\mathbb{R}^1$ which coincides with our intuition about open sets. Consider the collection

$$\tau \stackrel{\text{def}}{=} \left\{U \subseteq \mathbb{R}^1 \mid U \text{ is the union of open intervals}\right\}$$

where an open interval is defined as $(a, b) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^1 \mid a < x < b\}$ with $a, b \in \mathbb{R}^1$. This topology is called the classical or Hausdorff or Euclidean topology on $\mathbb{R}^1$.

By definition open intervals are in fact open subsets of $\mathbb{R}^1$. Other examples of open sets are $(a, +\infty) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^1 \mid a < x\}$ as can be seen from the description

$$(a, +\infty) = \bigcup_{n \in \mathbb{N}} (a, n) \ .$$

On the other hand, one can see that closed intervals $[a, b] \stackrel{\text{def}}{=} \{x \in \mathbb{R}^1 \mid a \leq x \leq b\}$ are indeed closed subsets of $\mathbb{R}^1$.

Exercise 1.8. *Decide if $[0, 1] \subseteq \mathbb{R}^1$ is an open subset in the classical topology.*

Example 1.9 (Euclidean spaces)**.** This example generalizes the real line to higher dimensions. Let $X = \mathbb{R}^n$, where $\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\}$ is the set of vectors with n real coordinates. We define the *open ball* with center $x \in \mathbb{R}^n$ and radius $\epsilon > 0$ as

$$\mathbb{B}(x, \epsilon) \stackrel{\text{def}}{=} \{y \in \mathbb{R}^n \mid |x - y| < \epsilon\} \ ,$$

where $|x| \stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^n x_i^2}$. The so-called *Euclidean* or *classical* or *Hausdorff* topology on $\mathbb{R}^n$ is given by the collection of arbitrary unions of open balls. More formally, a subset $U \in \mathbb{R}^n$ is open in the Euclidean topology if and only if there exists a collection of open balls $\{\mathbb{B}(x_\alpha, \epsilon_\alpha) \mid \alpha \in I\}$ such that

$$U = \bigcup_{\alpha \in I} \{\mathbb{B}(x_\alpha, \epsilon_\alpha) \mid \alpha \in I\} \ .$$

Exercise 1.10. *Verify that $\mathbb{R}^n$ with the classical topology is indeed a topological space.*

The above definition of open sets with balls prompts the following generalization for metric spaces, a concept somewhere halfway between Euclidean spaces and general topological spaces. First, a reminder.

Definition 1.11. A set X equipped with a function $d : X \times X \to \mathbb{R}^{\geq 0}$ is called a *metric space* (and the function d a *metric* or *distance function*) provided the following holds.

(1) For every $x \in X$ we have $d(x,x) = 0$; if $d(x,y) = 0$ for $x, y \in X$ then $x = y$.
(2) (Symmetry) For every $x, y \in X$ we have $d(x,y) = d(y,x)$.
(3) (Triangle inequality) If $x, y, z \in X$ are arbitrary elements, then

$$d(x,y) \leq d(x,z) + d(z,y) \ .$$

Exercise 1.12. *Show that $(\mathbb{R}^n, d)$ with $d(x,y) = |x - y|$ is metric space.*

Definition 1.13. Let (X, d) be a metric space. The *open ball* in X with center $x \in X$ and radius $\epsilon > 0$ is

$$\mathbb{B}(x, \epsilon) \stackrel{\text{def}}{=} \{y \in X \mid d(y,x) < \epsilon\} \ .$$

The topology τ_d induced by d consists of arbitrary unions of open balls in X.

Remark 1.14. A subset $U \subseteq (X, d)$ is open if and only if for every $y \in U$ there exists $\epsilon > 0$ (depending on y) such that $\mathbb{B}(y, \epsilon) \subseteq U$.

Most of the examples of topological spaces we meet in everyday life are induced by metrics (such topological spaces are called *metrizable*); however, as we will see, not all topologies arise from metrics.

Example 1.15. Let (X, τ) be a discrete topological space. Consider the function

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

One can see quickly that (X, d) is a metric space, and the topology induced by d is exactly τ.

Exercise 1.16. *Let (X, d) be a metric space, and define*

$$d_1(x,y) \stackrel{\textit{def}}{=} \begin{cases} d(x,y) & \textit{if } d(x,y) \leq 1 \\ 1 & \textit{if } d(x,y) > 1 \ . \end{cases}$$

Show that (X, d') is also a metric space, moreover d, d_1 induce the same topology on X (we could replace 1 by any positive real number).

Example 1.17 (Finite complement topology)**.** Let X be an arbitrary set. The *finite complement topology* on X has $\emptyset$ and all subsets with a finite complement as open sets. Alternatively, the closed subsets with respect to the finite complement topology are X and all finite subsets.

For the next example, we will quickly review what a partially ordered set is. Let X be an arbitrary set, $\leq$ a relation on X (i.e. $\leq \subseteq X \times X$). The relation $\leq$ is called a *partial order*, and $(X, \leq)$ a *partially ordered set* if $\leq$ is reflexive, antisymmetric, and transitive; that is

(1) (Reflexivity) for every $x \in X$ we have $x \leq x$
(2) (Antisymmetry) $x \leq y$ and $y \leq x$ implies $x = y$
(3) (Transitivity) $x \leq y$ and $y \leq z$ implies $x \leq z$.

Example 1.18 (Order topology)**.** Let $(X, \leq)$ be a partially ordered set. For an element $a \in X$ consider the one-sided intervals $\{b \in X \mid a < b\}$ and $\{b \in X \mid b < a\}$. The order topology τ consists of all finite unions of such.

Let us now recall how continuity is defined in calculus. A function $f : \mathbb{R} \to \mathbb{R}$ is called continuous if for every $x \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x - x'| < \delta \quad \text{whenever} \quad |f(x) - f(x')| < \epsilon$$

for all $x' \in X$. The naive idea behind this notion is the points that are 'close' to each other get mapped to points that are 'close' to each other as well in some sense.

The definition generalizes to metric spaces with no change, however, to obtain a notion of continuity for topological spaces, we need a reformulation in terms of open sets only. To this end, we reconsider how continuity is defined for functions between metric spaces. Let $f : (X, d) \to (Y, d')$ be a function of metric spaces; fix $x \in X$ arbitrary. Then f is called continuous at x if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x' \in X$ one has

$$d'(f(x), f(x')) < \epsilon \quad \text{whenever} \quad d(x, x') < \delta \ .$$

To phrase this in the language of open balls, it is equivalent to require that for every $\epsilon > 0$ there exists $\delta > 0$ for which

$$f(\mathbb{B}_X(x, \delta)) \subseteq \mathbb{B}_Y(f(x), \epsilon) \ .$$

Proposition 1.19. *A function $f : X \to Y$ between two metric spaces is continuous if and only if for every open set $U \subseteq Y$ the inverse image $f^{-1}(U) \subseteq X$ is open as well.*

Proof. Assume first that $f : (X, d) \to (Y, d')$ is continuous, that is, f is continuous at every $x \in X$. Choose an open set $U \subseteq Y$ (in the topology induced by d', of course), let $x \in f^{-1}(U)$ be arbitrary. Since $f(x)$ is contained in the open set $U \subseteq Y$, there exists $\epsilon_x > 0$ such that $\mathbb{B}_Y(f(x), \epsilon_x) \subseteq U$. By the continuity of f at x, we can find $\delta_x > 0$ with the property that $f(\mathbb{B}_X(x, \delta_x) \subseteq \mathbb{B}_Y(f(x), \epsilon_x)$, and hence

$$\mathbb{B}_X(x, \delta_x) \subseteq f^{-1}(U) .$$

But then

$$f^{-1}(U) = \bigcup_{x \in f^{-1}(U)} \mathbb{B}_X(x, \delta_x) ,$$

and hence $f^{-1}(U) \subseteq X$ is open.

Conversely, let us assume that the inverse image of every open subset of Y under f is open in X. Fix a point $x \in X$, and $\epsilon > 0$. Then $\mathbb{B}_Y(f(x), \epsilon) \subseteq Y$ is open, hence so is $f^{-1}(\mathbb{B}_Y(f(x), \epsilon))$, which in addition contains x. This implies that $f^{-1}(\mathbb{B}_Y(f(x), \epsilon))$ contains an open ball with center x and some radius, which we can take to be δ. $\square$

The result above motivates the following fundamental definition.

Definition 1.20 (Continuity)**.** Let X,Y be topological spaces, $f : X \to Y$ an arbitrary function. Then f is said to be *continuous* if the inverse image of every open set in Y is open in X. More formally, for every $U \subseteq Y$ open, $f^{-1}(U) \subseteq X$ is open as well. A *map* of topological spaces is a continuous function.

As taking inverse images preserves complements (i.e. $f^{-1}(Y - Z) = X - f^{-1}(Z))$, the continuity of f can be characterized equally well in terms of closed subsets of X: a function $f : X \to Y$ is continuous exactly if $f^{-1}(F) \subseteq X$ is closed for every closed subset $Z \subseteq Y$.

Exercise 1.21. *Prove the following statements.*

(1) Constant functions (ie. $f : X \to Y$ with $f(X)$ consisting of exactly one element) are continuous.
(2) The identity function $f : (X, \tau) \to (X, \tau)$, $x \mapsto x$ is continuous for every topological space X.
(3) Let X, Y, Z be topological spaces. If the functions $f : X \to Y$ and $g : Y \to Z$ are continuous, then so is the composition $g \circ f : X \to Z$ given by $(g \circ f)(x) \stackrel{def}{=} g(f(x))$.

Exercise 1.22. *Show that every function from a discrete topological space is continuous. Analogously, verify that every function to a trivial topological space is continuous.*

Interestingly enough, our definition of continuity is 'global' in the sense that no reference is made to individual points of the spaces X and Y. In fact, as opposed to the usual definition of continuity of functions on the real line, it is somewhat more delicate — and is less important in general — to define continuity at a given point of a topological space. In order to find the right notion first we need to pin down what it means to be 'close to a given point'.

Definition 1.23. Let (X, τ) be a topological space, $x \in X$ an arbitrary point. A subset $N \subseteq X$ is a *neighbourhood of* x if there exists an open set $U \subseteq x$ for which $x \in U \subseteq N$.

Remark 1.24. The terminology in the literature is ambiguous; it is often required that neighbourhoods be open. We call such a neighbourhood an *open neighbourhood*.

Remark 1.25. The intersection of two neighbourhoods of a given point is also a neighbourhood. If $U \subseteq X$ is an open set, then it is a neighbourhood of any of its points. In particular, X is a neighbourhood of every $x \in X$.

Remark 1.26. In a metric space X, a subset $N \subseteq X$ is a neighbourhood of a point $x \in X$ if and only if N contains an open ball centered at x.

Definition 1.27. Let (X, τ) be a topological space, $x \in X$. A collection $\mathcal{B}_x \subseteq P(X)$ of subsets all containing x is called a *neighbourhood basis of* x if

(1) every element of $\mathcal{B}_x$ is a neighbourhood of x;
(2) every neighbourhood of x contains an element of $\mathcal{B}_x$ as a subset.

Example 1.28. Let $X = \mathbb{R}^1$ be the real line with the Euclidean topology, $x = 0$. Then

$$\left\{\left(-\frac{1}{n}, \frac{1}{n}\right) \mid n \in \mathbb{N}\right\}$$

and

$$\left\{\left[-\frac{1}{n}, \frac{1}{n}\right] \mid n \in \mathbb{N}\right\}$$

are both neighbourhood bases of x. To put it in a more general context, let (X, d) be a metric space with the induced topology, $x \in X$ arbitrary. Then the collection

$$\left\{\mathbb{B}(x, \frac{1}{n}) \mid n \in \mathbb{N}\right\}$$

forms again a neighbourhood basis of $x \in X$.

Example 1.29 (Non-examples)**.** Consider again the case $X = \mathbb{R}^1$, $x = 0$. The collections of subsets below are *not* neighbourhood bases of x:

$$\left\{\left[0, \frac{1}{n}\right) \mid n \in \mathbb{N}\right\} \ , \ \{(-1, n) \mid n \in \mathbb{N}\} \ .$$

Beside their inherent usefulness neighbourhoods and neighbourhood bases serve the purpose letting us define the continuity of a function at a point.

Definition 1.30 (Continuity of a function at a point)**.** Let $f : X \to Y$ be a function between topological spaces, $x \in X$. We say that f *is continuous at the point* x, if for every neighbourhood N of $f(x)$ in Y there exists a neighbourhood M of x in X such that $f(M) \subseteq N$.

Remark 1.31. It is enough to require the condition in the definition for the elements of a neighbourhood basis of $f(x)$. To put it more clearly, let $\mathcal{B}_{f(x)}$ be a neighbourhood basis of $f(x)$ in Y. Then f as above is continuous at x if and only if for every $N \in \mathcal{B}_{f(x)}$ there exists a neighbourhood M of x in X with $f(M) \subseteq N$.

Remark 1.32. Observe that for an arbitrary map of sets $f : X \to Y$ and a subset $A \subseteq Y$ we have

$$f(f^{-1}(A)) = A \cap f(X) ,$$

hence $f(f^{-1}(A)) \subseteq A$. Therefore, if $f : X \to Y$ is a function between topological spaces, f is continuous at a point x if and only if for every neighbourhood N of $f(x)$ in Y, $f^{-1}(N)$ is a neighbourhood of $x \in X$.

Combining this observation with Remark 1.31, f is continuous at x precisely if for any neighbourhood basis $\mathcal{B}_{f(x)}$ of $f(x)$ in Y, the collection

$$\left\{ f^{-1}(N) \,|\, N \in \mathcal{B}_{f(x)} \right\}$$

is a neighbourhood basis of x.

The next result is our first theorem; note that as opposed to calculus, it is no longer the definition of continuity, but rather something we need to prove.

Theorem 1.33. *Let $f : X \to Y$ be a function between topological spaces. Then f is continuous if and only if it is continuous at x for every $x \in X$*

Proof. Assume first that $f : X \to Y$ is continuous, that is, the inverse image of every open set in Y under f is open in X. Fix a point $x \in X$; we will show that f is continuous at x. Let N be a neighbourhood of $f(x) \in Y$; this means that there exists an open set $V \subseteq Y$ for which $f(x) \in V \subseteq N$. By the continuity of f, $f^{-1}(V) \subseteq X$ is open, moreover

$$x \in f^{-1}(V) \subseteq f^{-1}(N) ,$$

hence $f^{-1}(N)$ is a neighbourhood of $x \in X$, and we are done by Remark 1.32.

To prove the other implication, assume that f is continuous at every $x \in X$; let $V \subseteq Y$ be an arbitrary open set. For any $x \in f^{-1}(V)$, the set V is a neighbourhood of $f(x)$, therefore $f^{-1}(V)$ is a neighbourhood of x since f is continuous at x. This means that for every $x \in f^{-1}(V)$ there exists an open subset $U_x \subseteq X$ for which $x \in U_x \subseteq f^{-1}(V)$. But then $f^{-1}(V) \subseteq X$ is open as

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x ,$$

i.e. it can be written as a union of open sets. □

The definition below is fundamental for the whole of topology.

Definition 1.34. A function $f : X \to Y$ between topological spaces is said to be a *homeomorphism*, if it is bijective, and both f and f^{-1} are continuous. Two topological spaces X and Y are called *homeomorphic* is there exists a homeomorphism from one to the other. This relation is denoted by $X \approx Y$.

Remark 1.35. The identitiy map $\mathrm{id} : X \to X$ is a homeomorphism. If $f : X \to Y$ is a homeomorphism then so is $f^{-1} : Y \to X$. If $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms, then so is $g \circ f : X \to Z$.

This implies that the relation 'being homeomorphic' is reflexive, symmetric, and transitive, hence an equivalence relation.

Exercise 1.36. *Fill in the details in Remark 1.35.*

Example 1.37. Note that a bijective continuous function is not necessarily a homeomorphism. We will see many examples of this phenomenon later, here are two simple ones. First, one can quickly check that if X is a trivial topological space (i.e. the only open sets in X are $\emptyset$ and X itself) then every function from every topological space to X is continuous.

Consider now a topological space (X, τ) where τ is not the trivial topology. Then the identity function $\mathrm{id}\colon (X, \{\emptyset, X\}) \to (X, \tau)$ is not continuous.

An analogous construction follows from the fact that every function *from* a discrete topological space to an arbitrary topological space is continuous.

If two topological spaces are homeomorphic, then not only their respective sets of points, but also their collections of open sets are in a one-to-one correspondence. Homeomorphisms show us when two topological spaces should be considered to be the same in the eye of topology. More precisely, we cannot distinguish homeomorphic topological spaces based on their topological structure.

In general it is not easy to show that two topological spaces are homeomorphic to each other; however, it can be equally difficult to prove that two topological spaces are *not* homeomorphic. We will see various methods both simple and hard that help us with such questions.

For now, let us get back to our investigation of open sets. As there can be many more open sets than we can easily handle in a random topological space, it is often very useful to come up with a small selection of open subsets that determine the whole topology.

Definition 1.38. Let (X, τ) be a topological space, $\mathcal{B} \subseteq P(X)$. The collection $\mathcal{B}$ is called a *basis for the topology* τ, if the open sets in X are precisely the unions of sets in $\mathcal{B}$.

A collection $\mathcal{S} \subseteq P(X)$ is called a *subbasis for the topology* τ if the set $\mathcal{B}(\mathcal{S})$ consisting of finite intersections of elements of $\mathcal{S}$ forms a basis for τ.

As the exercise below shows, if X is an arbitrary set, then any collection of subsets $\mathcal{S} \subseteq P(X)$ is a subbasis for *some* topology on X.

Exercise 1.39. *Let $\mathcal{S} \subseteq P(X)$ be an arbitrary set of subsets of X; define τ as the collection of arbitrary unions of finite intersections of elements of $\mathcal{S}$. Prove that τ is a topology on X. Also, show that if τ' is a topology on X such that $\mathcal{S} \subseteq \tau'$, then $\tau \subseteq \tau'$.*

The topology defined above is called the *topology generated by* $\mathcal{S}$. It is the smallest (with respect to inclusion of subsets of $P(X)$) topology where the elements of $\mathcal{S}$ are open. Note that the topology generated by a collection $\mathcal{S}$ might contain many more open sets than the elements of $\mathcal{S}$ (in fact often it contains many more than one would expect).

Example 1.40. Let $X = \{1, 2, 3\}$ be a set with just three elements. We will consider various sets of subsets, and calculate the corresponding generated topologies. First take $\mathcal{S} = \{\{1\}\}$. Then the set of finite intersections is $\mathcal{B}(\mathcal{S}) = \{X, \{1\}\}$, hence the topology generated by $\mathcal{S}$ (that is, the collection of arbitrary unions of elements of $\mathcal{B}(\mathcal{S})$) is $\{\emptyset, X, \{1\}\}$.

Next, choose $\mathcal{S} = \{\{1, 2\}, \{2, 3\}\}$. Then the set of finite intersections is $\mathcal{B}(\mathcal{S}) = \{X, \{1, 2\}, \{2, 3\}, \{2\}\}$, hence the generated topology is $\{\emptyset, X, \{1, 2\}, \{2, 3\}, \{2\}\}$.

Exercise 1.41. *Show that $\mathcal{S} = \{\{x\} \mid x \in X\}$ generates the discrete topology on an arbitrary set X.*

Exercise 1.42. *How many pairwise non-homeomorphic topologies are there on the set $X = \{1, 2, 3\}$?*

Example 1.43. Let $X = \mathbb{R}^1$, and $\mathcal{S} = \left\{\left(\frac{p}{q}, \frac{r}{s}\right) \mid p, q, r, s \in \mathbb{Z},\ q, s \neq 0\right\}$. Prove that $\mathcal{S}$ generates the Euclidean topology on $\mathbb{R}^1$.

There are several ways to measure how 'large' a topological space is. Here is a pair of notions based on the cardinality of sets.

Definition 1.44. A topological space (X, τ) is called *first countable*, if every point $x \in X$ has a countable neighbourhood basis. The topological space (X, τ) is *second countable*, if τ has a countable basis.

Exercise 1.45. *Is a discrete topological space first countable? Second countable?*

Exercise 1.46. *Does second countability imply first countability?*

Example 1.47. Euclidean spaces are second countable. The following collection gives a countable basis

$$\left\{\mathbb{B}(x, \frac{1}{m}) \mid x \in \mathbb{Q}^n,\ m \in \mathbb{N}\right\} .$$

Example 1.48. As evidenced by the collection

$$\left\{\mathbb{B}(x, \frac{1}{m}) \mid m \in \mathbb{N}\right\}$$

every metric space is first countable.

However, not every metric space is second countable. As an example we can take any uncountable set ($X = \mathbb{R}$ for instance) with the discrete topology. We have seen earlier that the discrete topology is induced by a metric. In this topology every singleton set $\{x\}$ is open, hence they need to belong to any basis for the topology; however there are uncountably many such sets.

The next two notions we will only need later; this is nevertheless a good place to introduce them.

Definition 1.49. Let $f : X \to Y$ be a function between topological spaces; f is called *open* if for every open set $U \subseteq X$ the image $f(U) \subseteq Y$ is open as well.

We can define closed functions in a completely analogous fashion.

Remark 1.50. Note that being open or closed is *not* the same as being continuous. It is true however that every homeomorphism is open and closed at the same time.

Later we will see examples that show that an open map is not necessarily closed, and vice versa.

Exercise 1.51. *Come up with a definition for convergence and Cauchy sequences in metric spaces.*

Exercise 1.52. (i) *Show that the functions $s, p : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by*

$$\begin{aligned} s(x,y) &= x+y \\ p(x,y) &= xy \end{aligned}$$

are continuous.
(ii) *Let $f, g : X \longrightarrow \mathbb{R}$ be continuous functions. Then all of $f \pm g$, $f \cdot g$ are continuous; if $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is continuous as well.*

Exercise 1.53. (i) *Is the function $f : \mathbb{R}^2 - \{(0,0)\} \longrightarrow \mathbb{R}^2$*

$$f(x,y) = \left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2} \right)$$

continuous on $\mathbb{R}^2 - \{(0,0)\}$?
(ii) *Is there a continuous function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ for which*

$$g|_{\mathbb{R}^2 - \{(0,0)\}} = f \ ?$$

Exercise 1.54. *Let α, β, γ be arbitrary real numbers. Then the so-called* open half-space

$$H = \left\{ (x,y,z) \in \mathbb{R}^3 \mid \alpha x + \beta y + \gamma z > 0 \right\}$$

is indeed open.

Exercise 1.55. *Prove that the set*

$$\{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \geq 10\}$$

is closed.

Exercise 1.56. *Is the set consisting of all point of the form $\frac{1}{n}$, n a natural number, open/closed in $\mathbb{R}$?*

Exercise 1.57. *Give examples of infinitely many open sets in $\mathbb{R}$, the intersection of which is (i) open (ii) closed (iii) neither open nor closed.*

Exercise 1.58. *Show that the closed ball*

$$D(x.\delta) = \{y \in \mathbb{R}^n \,|\, |x-y| \leq \delta\}$$

is indeed a closed subset of $\mathbb{R}^n$.

Exercise 1.59. *Prove that*

$$d_1(f,g) \;=\; \int_{[a,b]} |f-g|dx$$

is a metric on $\mathcal{C}[a,b]$. Is this still true if we replace continuous functions by Riemann integrable ones?

Definition 1.60. Let $p \in \mathbb{Z}$ be a fixed prime number. For an arbitrary nonzero integer $x \in \mathbb{Z}$ let

$$\mathrm{ord}_p(x) \stackrel{\mathrm{def}}{=} \text{ the highest power of } p \text{ which divides } x \ ,$$

while we define $\mathrm{ord}_p(0) \stackrel{\mathrm{def}}{=} \infty$.

If $\alpha = \dfrac{x}{y} \in \mathbb{Q}^\times$, then we set

$$\mathrm{ord}_p(\alpha) \;=\; \mathrm{ord}_p(\frac{x}{y}) \stackrel{\mathrm{def}}{=} \mathrm{ord}_p(x) - \mathrm{ord}_p(y) \ .$$

Note that the $\mathrm{ord}_p(\alpha)$ does not depend on the choice of x and y.

Exercise 1.61. *Show that for every $x, y \in \mathbb{Q}$*

(1) $ord_p(xy) = ord_p(x) + ord_p(y)$
(2) $ord_p(x+y) \geq \min\{ord_p(x), ord_p(y)\}$ *with equality if* $ord_p(x) \neq ord_p(y)$.

Compute the p-adic order of $5, 100, 24, -\frac{1}{48}, -\frac{12}{28}$ *for* $p = 2, 3, 5$.

Definition 1.62. With notation as so far, let $\alpha, \beta \in \mathbb{Q}$. Then we set

$$d_p(\alpha,\beta) \stackrel{\mathrm{def}}{=} \begin{cases} 0 & \text{if } \alpha = \beta \ , \\ \frac{1}{p^{\mathrm{ord}_p(\alpha-\beta)}} & \text{otherwise.} \end{cases}$$

This is called the *p-adic distance of α and β.*

Exercise 1.63. *Prove that* $(\mathbb{Q}, d_p)$ *is a metric space, which is in addition* non-archimedean*, that is, for every* $x, y, z \in \mathbb{Q}$ *one has*

$$d_p(x,y) \leq \max\{d_p(x,z), d_p(z,y)\} \ .$$

Conclude that in $(\mathbb{Q}, d_p)$ *every triangle is isosceles.*

Exercise 1.64. *Let* $f : X \to Y$ *be a continuous map. If* X *is first/second countable, then so is* $f(X)$.

Exercise 1.65. *Let* $f : X \to Y$ *be a function between topological spaces. Show that* f *is continuous if and only if*

$$f(\overline{A}) \subseteq \overline{f(A)}$$

for every subset $A \subseteq X$.

Exercise 1.66. *If* $f : X \to Y$ *is a continuous surjective open map, then* $F \subseteq Y$ *is closed exactly if* $f^{-1}(F) \subseteq X$ *is closed.*

2. Constructing topologies

2.1. **Subspace topology.** In this section we will start manufacturing new topologies out of old ones. There are various ways to do this, first we discuss topologies induced on subsets.

Definition 2.1. Let X be a topological space, $A \subseteq X$ an arbitrary subset. The *relative* or *subspace topology* on A is the collection of intersections with open sets in X.

In other words, a subset $U \subseteq A$ is open in the subspace topology if and only if there exists an open subset $V \subset X$ such that $U = V \cap A$.

Notation 2.2. To facilitate discussion and formalize the above definition, we introduce some notation. Let (X, τ) be a topological space, $A \subseteq X$ a subset. We denote the subspace topology on A by

$$\tau_A \overset{\text{def}}{=} \{A \cap V \mid V \in \tau\} \ .$$

Remark 2.3. Here is another way of thinking about the subspace topology. Let (X, τ) be a topological space, $A \subseteq X$ a subset, $i : A \hookrightarrow X$ the inclusion function. Then τ_A is the smallest topology which makes i continuous.

The following is a related notion, which will play an important role in later developments.

Definition 2.4. A *pair* is an ordered pair (X, A), with X a topological space, and $A \subseteq X$ an arbitrary subset equipped with the subspace topology. A continuous map of pairs $f : (X, A) \to (Y, B)$ is a continuous map $f : X \to Y$ for which $f(A) \subseteq B$.

Example 2.5. Let $[0,1] \subseteq \mathbb{R}$. Open subsets in $\mathbb{R}$ are unions of open intervals. Therefore elements of τ_A are arbitrary unions of sets of the form $[0,1] \cap (a,b)$ with $a, b \in \mathbb{R}$. For example $[0, \frac{1}{2}) = [0,1] \cap (\frac{1}{2}, 2)$ is open in $[0,1]$.

Remark 2.6. A subset $U \subseteq A \subseteq X$ which is open in the subset topology in A will typically not be open in X.

Exercise 2.7. *Prove the following statements.*

(1) If $f : X \to Y$ is a map, $A \subseteq X$ a subspace, then

$$f|_A : A \longrightarrow Y$$

given by $f|_A(x) = f(x)$ whenever $x \in A$, is a continuous function.

(2) Let X, Y be topological space, $Y_1 \subseteq Y_2 \subseteq Y$ subspaces, $f : X \to Y_2$ a continuous function. Then f as a function $X \to Y$ is also continuous. If $f(X) \subseteq Y_1$, then f as a function from X to Y_1 is continuous as well.

Proposition 2.8. *Let $(X, \tau), (Y, \tau')$ be topological spaces, $A, B \subseteq X$ closed subsets such that $X = A \cup B$. Assume we are given continuous functions $f : (A, \tau_A) \to Y$, $g : (B, \tau_B) \to Y$ such that*

$$f|_{A\cap B} = g|_{A\cap B} .$$

Then there exists a unique continuous function $h : X \to Y$ for which

$$h|_A = f \quad \text{and} \quad h|_B = g .$$

Proof. Set

$$h(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B . \end{cases}$$

This gives a well-defined function $h : X \to Y$ by assumption.

Let $F \subseteq Y$ be a closed subset. Observe that

$$h^{-1}(F) = f^{-1}(F) \cup g^{-1}(F) .$$

Because f is continuous, $f^{-1}(F) \subseteq A$ is closed, but then $f^{-1}(F) \subseteq X$ is also closed, since A was closed in X. By the same argument, $g^{-1}(F) \subseteq X$ is closed, too. But then $h^{-1}(F) \subseteq X$ is again closed. □

With the help of the subspace topology we will generalize the familiar notions of the interior, boundary, and closure of a subset.

Proposition 2.9. *Let (X, τ) be a topological space, $A \subseteq X$ an arbitrary subset.*

(1) there exists a largest (with respect to inclusion) open (in X) set $U \subseteq A$. This is called the interior of A, *and denoted by* $\text{int}_X(A)$, $\text{int}(A)$, *or* A°.

(2) There exists a smallest (with respect to inclusion) closed (in X) subset $F \supseteq A$. This is called the closure of A, *and denoted by* $\overline{A}^X$ *or simply* $\overline{A}$.

(3) If $A \subseteq Y \subseteq X$, then

$$\overline{A}^Y = \overline{A}^X \cap Y \ .$$

If $Y \subseteq X$ is closed, then $\overline{A}^Y = \overline{A}^X$.

Proof. (1) Consider the union of all subsets $U \subseteq A$ which are open in X. This is by construction the largest open (in X) subset contained in A, and is uniquely determined.

(2) Analogously to the previous case, the closure of A is the intersection of all subsets $F \supseteq A$ that are closed in X. Again, by construction it is uniquely determined.

(3) Based on the construction of the closure of A in X we have

$$\begin{aligned} \overline{A}^X \cap Y &= \Big(\bigcap_{A\subseteq F, X-A\in\tau} F\Big) \cap Y \\ &= \bigcap_{A\subseteq F, X-A\in\tau} (F\cap Y) \\ &= \bigcap_{A\subseteq F\cap Y, Y-F\in\tau_Y} F\cap Y \\ &= \overline{A}^Y \end{aligned}$$

due to the definition of the subspace topology on Y. If $Y \subseteq X$ is closed, then $\overline{A}^X \subseteq Y$, hence the result.

□

Exercise 2.10. *In the situation of the Proposition, show that a point $x \in X$ lies in the closure of A if and only if every open neighbourhood of x in X intersects A.*

Remark 2.11. Even if $A \subseteq X$ seems naively relatively large (think $\mathbb{Q} \subseteq \mathbb{R}$ for example) it can happen that $\operatorname{int} A = \emptyset$. In a similar vein, although $\mathbb{Q} \subseteq \mathbb{R}$ is in a way smaller, we have $\overline{\mathbb{Q}} = \mathbb{R}$.

Exercise 2.12. *Prove that $\overline{A}^X = X - \operatorname{int}_X(X - A)$.*

Proposition 2.13. *With notation as above, let $Y \subseteq X$, $\mathcal{B}$ a basis for the topology τ. Then*

$$\mathcal{B}_Y \stackrel{def}{=} \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis of the subspace topology τ_Y.

In a similar fashion, if $x \in Y \subseteq X$ is an arbitrary point, $\mathcal{B}_x \subseteq \tau$ a neighbourhood basis of x, then $(\mathcal{B}_x)_Y \stackrel{def}{=} \{N \cap Y \mid N \in \mathcal{B}_x\}$ is a neighbourhood basis of x in τ_Y.

Proof. Left as an exercise. □

Definition 2.14. Let again be (X, τ) a topological space, $A \subseteq X$ a subset. The *boundary of* A denoted by ∂A is defined as

$$\partial A \stackrel{\text{def}}{=} \overline{A} \cap \overline{X - A} \ .$$

Exercise 2.15. *Show that* $\partial A = \overline{A} - \operatorname{int}_X A$.

Definition 2.16. A subset $A \subseteq X$ of a topological space is called *dense*, if $\overline{A} = X$. A subset $A \subseteq X$ is callled *nowhere dense*, if $\operatorname{int} \overline{A} = \emptyset$.

As immediate examples observe that $\mathbb{Q} \subseteq \mathbb{R}$ is dense, while $\mathbb{Z} \subseteq \mathbb{R}$ is nowhere dense. Another simple situation is of course the discrete topology: in this case no subset different from X is dense, and no non-empty subset is nowhere dense.

Exercise 2.17. *Is there an uncountable nowhere dense set in* $[0, 1]$*?*

Definition 2.18 (Limit point)**.** Let (X, τ) be a topological space, $A \subseteq X$ arbitrary. A point $x \in X$ is a *limit point* or *accumulation point* or *cluster point* of A, if $x \in \overline{A - \{x\}}$.

A point $x \in A$ is an isolated point of A if there exists an open (in X) neighbourhood U of x for which $U \cap A = \emptyset$.

A limit point x of A may or may not lie in A.

Remark 2.19. Note that a point x is a limit point of A if and only if every neighbourhood of x intersects A in a point different from x.

Exercise 2.20. *List all limit points of the following sets:* $(0, 1] \subseteq \mathbb{R}$,$\left\{\frac{1}{n} \,|\, n \in \mathbb{N}\right\} \subseteq \mathbb{R}$,$(0, 1) \cup \{3\} \subseteq \mathbb{R}$, $\mathbb{Q} \subseteq \mathbb{R}$.

It is intuitively plausible that there is a close relation between the closure of a subset and its limit points.

Proposition 2.21. *Let* (X, τ) *be a topological space,* $A \subseteq X$ *a subset, denote* A' *the set of limit points of* A*. Then*

$$\overline{A} \ = \ A \cup A' \ .$$

Proof. First we prove that $A \cup A' \subseteq \overline{A}$. The containment $A \subseteq \overline{A}$ is definitional. Let $x \in A'$. Then every neighbourhood of x intersects A, hence $x \in \overline{A}$.

For the other direction, let $x \in \overline{A} - A$. As $x \in \overline{A}$, every open neighbourhood intersects A, as $x \notin A$, the intersection point must be a point of A other than x. Therefore $x \in A'$. $\square$

Corollary 2.22. *A subset* $A \subseteq X$ *is closed if and only if* A *contains all of its limit points.*

A fundamental and closely related notion is the convergence of sequences. Since in general it behaves rather erraticly and certainly not according to our intuition trained in Euclidean spaces, it is rarely discussed in this generality, in spite of the fact that there is nothing complicated about it.

Definition 2.23. Let (X, τ) be a topological space, (x_n) a sequence of points in X, $x \in X$ arbitrary. We say that the *sequence x_n converges to x* if for every neighbourhood B of x there exists a natural number M_B such that $x_n \in B$ whenever $n \geq M_B$. This fact is denoted by $x_n \to x$.

The limit point of a subset of a topological space and the limit of a convergent sequence are different (although admittedly closely related) notions, and one should exercise caution not to confuse them.

It is routine to check using the neighbourhood basis of a point consisting of open balls that the above definition is equivalent to the usual one in a metric space. It is very important to point out that in a general topological space the limit of a convergent sequence is *not* unique. One reason for this is that if $U \subseteq X$ is a minimal open set (i.e. it contains no other non-empty open sets) and $x \in U$ is the limit of a sequence (x_n), then so is any other element $y \in U$. In extreme cases a sequence may converge to all points of the given topological space X (this happens for example in a trivial topological space, where every sequence of points converges to every point).

Worse, continuity can no longer be characterized with the help of convergent sequences.

Example 2.24 (Non-uniqueness of limits of sequences)**.** Let $X = \{1, 2, 3\}$, $\tau = \emptyset, \{1, 2\}, X$. It is easy to see using the definition that all sequences in X converge to 3, while sequences with eventually only 1's and 2's in them converge to 1,2, and 3.

Exercise 2.25. *Let X be a second countable topological space, $A \subseteq X$ an uncountable set. Verify that uncountably many points of A are limit points of A.*

Exercise 2.26. *Prove that a subset $U \subseteq X$ is open if and only if $\partial U = \overline{U} \setminus U$.*

Exercise 2.27. *Prove that if a topological space X has a countable dense subset, then every collection of disjoint open subsets is countable.*

Exercise 2.28. *Let $f : X \longrightarrow Y$ be a homeomorphism, x_k a sequence in X. Then x_k is convergent in X if and only if $f(x_k)$ is convergent in Y.*

2.2. **Local properties.** As we have seen, forming the interior and/or closure behaves well with respect to taking complements. It can make life difficult, however, that the same cannot be said about taking intersections. Another problem one faces is that knowing the interior or the closure of subspace is not enough to 'reconstruct' the subspace itself. There are plenty of subsets $A \subseteq \mathbb{R}$ for example with empty interior and closure the whole of $\mathbb{R}$.

Proposition 2.29. *Let $U \subseteq X$ be an open subset, $A \subseteq \mathbb{R}$ arbitrary. Then*

$$\overline{A \cap U}^X \cap U = \overline{A}^X \cap U \ .$$

Corollary 2.30. *A subset $F \subseteq U$ is closed in U if and only if*

$$U \cap \overline{F \cap U}^X = F \cap U \ .$$

Exercise 2.31. *Prove that if $U \subseteq X$ is the interior of a closed subset $F \subseteq X$, then* $\operatorname{int}(\overline{U}) = U$.

Proof. We start with unwinding the definitions. On this note, observe that

$$\overline{A}^X \cap U = \left(\bigcap_{F \supseteq A, F \subseteq X \text{ closed}} F \right) \cap U = \bigcap_{F \supseteq A, F \subseteq X \text{ closed}} F \cap U \ ,$$

while

$$\overline{F \cap U}^X \cap U = \left(\bigcap_{F \supseteq A \cap U, F \subseteq X \text{ closed}} F \right) \cap U = \bigcap_{F \supseteq A \cap U, F \subseteq X \text{ closed}} F \cap U$$

What we need to prove now is that

$$\bigcap_{F \supseteq A, F \subseteq X \text{ closed}} F \cap U = \bigcap_{F \supseteq A \cap U, F \subseteq X \text{ closed}} F \cap U \ .$$

Since all closed sets F that occur on the left-hand side show up on the right-hand side as well,

$$\bigcap_{F \supseteq A, F \subseteq X \text{ closed}} F \cap U \supseteq \bigcap_{F \supseteq A \cap U, F \subseteq X \text{ closed}} F \cap U \tag{1}$$

is immediate.

Suppose now that there exists an element $x \in X$ such that

$$x \in \bigcap_{F \supseteq A, F \subseteq X \text{ closed}} F \cap U \quad \text{but} \quad x \notin \bigcap_{F \supseteq A \cap U, F \subseteq X \text{ closed}} F \cap U \ .$$

This means that there exists a closed subset $F' \subseteq X$ with $F' \supseteq A \cap U$ such that $x \notin F'$. As x belongs to the left-hand side of (1), $x \in U$, hence $x \in U - (F \cap U)$. Consider now the subset $G \stackrel{\text{def}}{=} (X - U) \cup (U \cap F') \subseteq X$. Being the complement of the open set $U - U \cap F'$ in X, G is closed, moreover $x \notin G$.

However, since $A = (A \cap U) \cup (A - U) \subseteq (F' \cap U) \cup (X - U) = G$, we have found a term on the left-hand side of (1) which does not contain x, a contradiction. ☐

Exercise 2.32. *Find examples of open subsets in $\mathbb{R}$ that are* not *the interiors of their closures.*

Taking this route a bit further, we arrive at the following result, which motivates the local study of topologies. A collection $\{U_\alpha \,|\, \alpha \in I\}$ of open subsets of X whose union is X is called an *open cover of X*.

Proposition 2.33. *Let X be a topological space, $A \subseteq X$ an arbitrary subspace, $(U_\alpha \,|\, \alpha \in I)$ an open cover of X. Then A is closed in X if and only if $A \cap U_\alpha$ is open in U_α for every $\alpha \in I$.*

Proof. The set A is closed in X if and only if $X - A$ is open in A. Since all the U_α's are open in X, $X - A$ is open in X if and only if $U_\alpha \cap (X - A)$ is open in U_α for every $\alpha \in I$. $\square$

We say that being closed is a *local property*, as it can be tested on some (any) open cover of X. Note that it is very important that we take a cover of X, and not one of A.

Definition 2.34. A subset $A \subseteq X$ is called *locally closed in X*, if every $a \in A$ has an open neighbourhood $U_a \in X$ such that $A \cap U_a$ is closed in U_a.

Example 2.35. The subset $(-1, 1) \times \{0\} \subseteq \mathbb{R}^2$ is locally closed.

Note that every open set is locally closed in any topological space by definition. Luckily it turns out that the structure of locally closed subsets is actually simpler than one might guess.

Proposition 2.36. *A subspace $A \subseteq X$ of a topological space is locally closed if and only if $A = F \cap U$, where $F \subseteq X$ is closed, and $U \subseteq X$ is open.*

Proof. Assume first that A has the shape $A = F \cap U$, with F closed, and U open in X. Then we can verify that A is locally closed immediately by taking $U_a \stackrel{\text{def}}{=} U$ for all $a \in A$ in the definition of local closedness.

Conversely, let A be locally closed, and $(U_a \,|\, a \in A)$ an collection of suitable open subsets of X. Set $U \stackrel{\text{def}}{=} \cup_{a \in A} U_a$. Then $A \subseteq U$, and the fact that being closed is a local property does the trick. $\square$

Proposition 2.37. *Let $f : X \to Y$ be a function between topological spaces, $\mathfrak{U} \subseteq \tau_X$ a collection, such that the union of some of its elements equals X. Then f is continuous if and only if $f|_{U_\alpha} : U_\alpha \to Y$ is continuous for every $U_\alpha \in \mathfrak{U}$.*

Proof. To come. $\square$

2.3. **Product topology.** We are looking for a way to put a topology on the Cartesian product $X \times Y$ of two sets that is in a way 'natural'. The product of the two sets does not come alone, but with two projection functions $\pi_X : X \times Y \to X$, $\pi_X(x, y) = x$, and $\pi_Y : X \times Y \to Y$, $\pi_Y(x, y) = y$.

$$\begin{tikzcd}
& X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
X & & Y
\end{tikzcd}$$

We will require that both projection functions pr_X and pr_Y be continuous. As we will see, this defines a unique topology on the Cartesian product, the minimal (containing the least number of open sets) for which the continuity of the projections holds.

Let us have a look, which open sets are needed. Let $U \subseteq X$ be an open set. Then

$$\pi_X^{-1}(U) = U \times Y \subseteq X \times Y$$

has to be open, and analogously for open subsets $V \subseteq Y$. Moreover, since

$$\pi_X^{-1}(U) \cap \pi_Y^{-1}(V) = (U \times Y) \cap (X \times V) = U \times V ,$$

all subsets of the form $U \times V$ are open when $U \subseteq X$ and $V \subseteq Y$ are open. It is quickly checked that the intersection of such sets is again of the same form, and hence form the basis of a topology.

Definition 2.38. Let X, Y be topological spaces. The *product topology* on the set $X \times Y$ consists of arbitrary unions of subsets of the form $U \times V$, with $U \subseteq X$,$V \subseteq Y$ open.

It is important to point out that *not* all subsets of the product topology are products of open sets. The same definition holds for the product of finitely many topological spaces. All our results for two topological spaces will hold for arbitrary finite products. However, we will mostly content ourselves with the case of two spaces for ease of notation.

Remark 2.39. We can define analogously the product of infinitely many topological spaces. If $\{X_\alpha \,|\, \alpha \in I\}$ is a collection of topological spaces, then the product topology on $\times_{\alpha \in I} X_\alpha$ is given by the basis of open sets

$$\times_{\alpha \in I} U_\alpha$$

where $U_\alpha \subseteq X_\alpha$ are open for all $\alpha \in I$, and $U_\alpha = X_\alpha$ for all but finitely many α's.

Lemma 2.40. *Let (X, τ_X) and (Y, τ_Y) be topological spaces, $\mathcal{U}$ and $\mathcal{V}$ bases for τ_X and τ_Y, respectively. Then the collection of sets*

$$\mathcal{W} \overset{def}{=} \{S \times T \,|\, S \in \mathcal{U}, T \in \mathcal{V}\}$$

forms a basis of the product topology on $X \times Y$.

Proof. Let $U \in \tau_X$, $V \in \tau_Y$ be arbitrary open sets. It is then enough to prove that $U \times V$ can be written as a union of elements of $\mathcal{W}$. To this end, write

$$U = \bigcup_{\alpha \in I} U_\alpha \ , \ V = \bigcup_{\beta \in J} V_\beta \ ,$$

where $U_\alpha \in \mathcal{U}$, and $V_\beta \in \mathcal{V}$ for every $\alpha \in I, \beta \in J$. Then

$$U \times V = \bigcup_{\alpha \in I, \beta \in J} U_\alpha \times V_\beta \ ,$$

a union of elements of $\mathcal{W}$. $\square$

Remark 2.41. Observe whenever we have topological spaces $(X, \tau_X), (Y, \tau_Y)$ and subsets $A \subseteq X, B \subseteq Y$, then there are two natural ways to put a topology on $A \times B$. Namely, we can first take the subspace topologies on A and B induced from that of X and Y, respectively, and then form the product space of A and B. Or, we can consider $A \times B$ as a subset of $X \times Y$ and equip it with the subspace topology coming from $X \times Y$. The way out of this apparently unfortunate situation is that these two constructions produce the same result.

This can be seen as follows. We will denote the topology on $A \times B$ by first taking subspaces and then products by τ_1, and the topology obtained by taking the product first and then the subspace topology by τ_2. Now we show that the collection

$$\mathcal{U} \stackrel{\text{def}}{=} \{(U \cap A) \times (V \cap B) \,|\, U \in \tau_X, V \in \tau_Y\}$$

is a basis for both τ_1 and τ_2, hence $\tau_1 = \tau_2$.

Let $U \subseteq X, V \subseteq Y$ be open sets. Since such subsets form a basis for the topology of $X \times Y$, subsets of the form $(U \times V) \cap (A \times B)$ will provide a basis of the subspace topology on $A \times B$ inherited from $X \times Y$. However,

$$(U \times V) \cap (A \times B) \;=\; (U \cap A) \times (V \times B) \;,$$

thus proving that $\mathcal{U}$ is a basis for τ_1. On the other hand,

$$\{U \cap A \,|\, U \in \tau_X\} \;,\; \{V \cap B \,|\, V \in \tau_Y\}$$

are bases for $(\tau_X)|_A$ and $(\tau_Y)|_B$, respectively. Since the products of two bases form a basis for the product topology, $\mathcal{U}$ is indeed a basis for τ_2.

The following result characterizes products up to a unique homeomorphism. The proof works with minimal changes in the infinite case, but we will only deal with the finite case now.

Proposition 2.42. *Let X and Y be topological spaces. Then the product space $X \times Y$ has the following property: for every topological space Z with continuous maps $f_X : Z \to X$ and $f_Y : Z \to Y$ there exists a unique map $g : Z \to X \times Y$ such that the following diagram commutes.*

$$\begin{tikzcd}
Z \arrow[dr, "g"] \arrow[drr, "f_Y"] \arrow[ddr, "f_X"'] & & \\
& X \times Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \\
& X &
\end{tikzcd}$$

In addition, $X \times Y$ equipped with the product topology is the only such space up to a unique homeomorphism.

The commutativity of the diagram means that $\pi_X \circ g = f_X$ and $\pi_Y \circ g = f_Y$.

Remark 2.43. By general algebra Proposition 2.42 implies that the product topology on $X \times Y$ is in fact unique up to a unique homeomorphism.

Proof of Proposition 2.42. Let Z be a topological space as in the statement with maps f_X and f_Y. Then the continuity of the diagram determines uniquely the set-theoretic function $g : Z \to X \times Y$ via

$$g(z) = (f_X(z), f_Y(z)) .$$

We need to prove that g is continuous. To this end, fix an open set $U \subseteq X \times Y$. Then

$$U = \bigcup_{i=1}^{n} V_i \times W_i$$

with $V_i \subseteq X$ and $W_i \subseteq Y$ being open sets for every $1 \leq i \leq n$. Hence

$$\begin{aligned} g^{-1}(U) &= g^{-1}\left(\bigcup_{i=1}^{n} V_i \times W_i\right) \\ &= \bigcup_{i=1}^{n} g^{-1}(V_i \times W_i) \\ &= \bigcup_{i=1}^{n} \left(g^{-1}(\pi_X^{-1}(V_i) \cap \pi_Y^{-1}(W_i))\right) \\ &= \bigcup_{i=1}^{n} \left(f_X^{-1}(V_i) \cap f_Y^{-1}(W_i)\right) , \end{aligned}$$

which shows that $g^{-1}(U) \subseteq Z$ is indeed open.

□

Exercise 2.44 (Products of metric spaces)**.** *Let $N : \mathbb{R}^m \to \mathbb{R}$ be any norm with the property that it is monotonically increasing function in every coordinate while keeping all others fixed. Consider finitely many metric spaces $(X_1, \delta_1), \ldots, (X_m, d_m)$. Show that the function*

$$d_N((x_1, \ldots, x_m), (y_1, \ldots, y_m)) \stackrel{def}{=} N(d_1(x_1, y_1), \ldots, d_m(x_m, y_m))$$

defines a metric on the Cartesian product set $X_1 \times \cdots \times X_m$. Prove that all such metrics d_N are bounded by above and below by a positive multiple of each other, and each one gives rise to the product of the metric topologies on $X_1 \times \cdots \times X_m$.

Theorem 2.45. *Let X,Y and Z be topological spaces, $f : Z \to X \times Y$ be an arbitrary function. Let $f_X = \pi_X \circ f$ and $f_Y = \pi_Y \circ f$ (we will call them 'coordinate functions'). For any point $z \in Z$, the function f is continuous at z if and only if both functions f_X and f_Y are continuous at z.*

Proof. If f is continuous, then so are $f_X = f \circ \pi_X$ and $f_Y = f \circ \pi_Y$ since they are both compositions of continuous functions.

Assume now that f_X and f_Y are continuous. Then Proposition 2.42 implies the existence of a unique continuous map $g : Z \to X \times Y$ whose compositions with the appropriate projection maps are f_X and f_Y. But this implies $f = g$, and hence f must be continuous. □

Exercise 2.46. *Show that the projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y$ are open maps.*

Exercise 2.47. *Let $A \subseteq X, B \subseteq Y$ be subspaces of the indicated topological spaces. Verify that*

$$\overline{A \times B} = \overline{A} \times \overline{B}$$

inside $X \times Y$.

Exercise 2.48. *Let $f : X \to Y$ and $g : W \to Z$ be open maps. Show that $f \times g : X \times W \to Y \times Z$ is open as well.*

2.4. **Gluing topologies.** It happens often that one tries to construct topological spaces from small pieces, that somehow match and can be used to glue things together. We take on this method now systematically.

For starters, consider the following situation. Let X be a topological space, $\{U_\alpha \,|\, i \in I\}$ an open cover of X. For a subset $U \subseteq X$ to be open is a local property, that is, U is open in X if and only $U \cap U_\alpha$ is open in U_α for every $\alpha \in I$.

We can play a similar game with morphisms: continuing from the previous paragaph, let $f : X \to Y$ be a continuous map; by restricting to the elements of the open cover we obtain continuous maps $f_\alpha : U_\alpha \to Y$ that agree on the intersections, i.e.

$$f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$$

Conversely, assume we are given continuous maps $f_\alpha : U_\alpha \to Y$ for every $\alpha \in I$ that agree on the overlaps, i.e. $f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$, then they fit together to give a unique (set-theoretic) function $f : X \to Y$ satisfying $f|_{U_\alpha} = f_\alpha$ for every $\alpha \in I$; moreover this function f is continuous. To see why this is so, take any open set $V \subseteq Y$. Then $f^{-1}(V) \subseteq X$ is open, since $f^{-1}(V) \cap U_\alpha = f_\alpha^{-1}(V) \subseteq U_\alpha$ is open, and openness is a local property.

The moral of the story is that once there is an open cover $\{U_\alpha \,|\, i \in I\}$ of X given, one can view continuous maps $f : X \to Y$ as collections of continuous maps $f_\alpha : U_\alpha \to X$ that are compatible on the overlaps. Now we will run this procedure in reverse.

Theorem 2.49. *Let X be an arbitrary set, $\{X_\alpha \,|\, \alpha \in I\}$ a collection of subsets of X whose union is X. Assume that for each $\alpha \in I$ there is given a topology τ_α on X_α such that for every $\alpha, \beta \in I$ the subset $X_\alpha \cap X_\beta$ is open in both X_α and X_β; moreover, the induced topologies on $X_\alpha \cap X_\beta$ from τ_α and τ_β coincide.*

Then there is a unique topology τ on X inducing upon each X_α the topology τ_α.

Definition 2.50. We say that the topology τ constructed in Theorem 2.49 is obtained by *gluing* the topologies on the X_α's.

Note that it is not in general clear what properties of the X_α's get inherited by X. For example if we glue together a compatible collection of Hausdorff topological spaces, then the result will *not* be in general Hausdorff, as the following example shows.

Example 2.51. Let X be the real line with the origin counted twice, that is, $X = (\mathbb{R} - 0) \cup \{0_1\} \cup \{0_2\}$, $X_1 = (\mathbb{R} - \{0\}) \cup \{0_1\}$, $X_2 = (\mathbb{R} - \{0\}) \cup \{0_2\}$. Give both X_i's the topology of the real line, and X the one obtained by gluing X_1 and X_2 along their common open subset $\mathbb{R} - \{0_i\}$. Then the images of the two origins in X are different points, it is not however possible to separate them with disjoint open sets (any open set containing one of these points will contain an open interval around it).

Proof of Theorem 2.49. We deal with uniqueness first. Let τ be a topology on X inducing the τ_α's on the X_α's, and making all $X_\alpha \subseteq X$ open. Since the X_α's form an open cover of X, a subset $U \subseteq X$ is open if and only if $U \cap X_\alpha$ is open for the topology $\tau|_{X_\alpha}$ for every $\alpha \in I$, if and only if $U \cap X_\alpha$ is open in τ_α for every $\alpha \in I$. This last condition however only depends on the X_α's , hence uniqueness.

We will use the argument in reverse to construct τ. Define

$$\tau \stackrel{\text{def}}{=} \{U \subseteq X \,|\, U \cap X_\alpha \in \tau_\alpha \,\forall \alpha \in I\} \ .$$

First we show that τ is indeed a topology on X. Obviously, $\emptyset$ and X belong to τ. Let now $\{U_j \,|\, j \in J\}$ be an arbitrary collection of elements of τ, we want to prove that their union U lies again in τ. Thus, we need that $U \cap X_\alpha \subseteq X_\alpha$ is open. But

$$U \cap X_\alpha = \left(\bigcup \{U_j \,|\, j \in J\}\right) \cap X_\alpha = \bigcup_{j \in J} (U_j \cap X_\alpha) \ ,$$

hence $U \cap X_\alpha$ is open in X_α, since all intersections $U_j \cap X_\alpha$ are open in X_α by choice, and τ_α is a topology on X_α. A completely analogous reasoning takes care of finite intersections.

We are left with showing that τ indeed induces τ_α on X_α. Take a subset $U \subseteq X_\alpha$; we have to show that U is in τ_α if and only if it is in $\tau|_{X_\alpha}$. Since X_α is τ-open in X, it is enough to prove that $U \in \tau_\alpha$ if and only if $U \in \tau$.

By definition of τ, $U \in \tau$ if and only if $U \cap X_\beta \in \tau_\beta$ for every $\beta \in I$, in particular $U \cap X_\alpha = U$ is τ_α-open.

To settle the other direction, we need to prove that $U \cap X_\beta \subseteq X_\beta$ is τ_β-open for all $\beta \in I$. We are assuming that $X_\alpha \cap X_\beta$ inherits the same topology from both X_α and X_β, and it is open in each, therefore the subset $U \cap X_\beta \subseteq X_\alpha \cap X_\beta$ is open in X_β if and only if it is open for $\tau_\beta|_{X_\alpha \cap X_\beta}$. This latter topology is however the same as

$\tau_\alpha|_{X_\alpha\cap X_\beta}$ by the compatibility assumption on the τ_a's, so since $U\cap X_\beta = U\cap X_\alpha\cap X_\beta$, it follows that $U\cap(X_\alpha\cap X_\beta)$ is indeed open in $X_\alpha\cap X_\beta$ for the topology induced by τ_β. □

3. Connectedness

Here we formulate the mathematical version of the naive notion that a space is connected.

Definition 3.1. A topological space X is *connected* if it cannot be written as the union of two disjoint nonempty open sets. Otherwise X is called *disconnected*. A subset $A\subseteq X$ is called *clopen* if A is both open and closed.

Remark 3.2. The complement of a clopen subset is also clopen. X is connected if and only if the only clopen subsets in X are the empty set and itself.

Definition 3.3. A *discrete-valued map* is a map $f : X \to D$ with D a discrete topological space.

Proposition 3.4. *A topological space X is connected if and only if every discrete-valued map on X is constant.*

Proof. Assume first that X is connected. For every $d\in D$, the set $\{d\}\subseteq D$ is clopen, hence so is its inverse image $f^{-1}(d)\subseteq X$. Therefore $f^{-1}(d)$ is either empty or the whole of X. As these are pairwise disjoint, there is exactly one d for which $f^{-1}(d)=X$; hence f is constant.

Next, assume that every discrete-valued map from X is constant. Suppose that X is disconnected, that is, $X=U\cup V$ the union of two disjoint clopen sets. Then the function

$$f(x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x\in U \\ 1 & \text{if } x\in V\ . \end{cases}$$

is discrete-valued and continuous, a contradiction. □

Proposition 3.5. *If $f : X \to Y$ is a continuous function of topological spaces and X is connected, then so is $f(X)$.*

Proof. We will show that every discrete-valued map $d : f(X)\to D$ is constant. Pick such a map d, then the composition $d\circ f : X \to D$ is a discrete valued map from X, hence constant. As f is surjective onto $f(X)$, d must be constant as well. This means that $f(X)$ is connected. □

Proposition 3.6. *If $\{Y_i\,|\,i\in I\}$ is a collection of connected subsets in a topological space X (all equipped with the subspace topology), and no two of the Y_i's are disjoint, then $\cup_{i\in I} Y_i$ is connected.*

Proof. Let $d : \cup_{i\in I} Y_i \to D$ be a discrete-valued map. Fix two arbitrary points $p, q \in \cup_{i\in I} Y_i$. Without loss of generality we can assume that $p \in Y_1$ and $q \in Y_2$. Pick an arbitrary point $r \in Y_1 \cap Y_2 \neq \emptyset$. As d is constant on both Y_1 and Y_2, $d(p) = d(r) = d(q)$. Since the points p and q were chosen in an arbitrary way, d is constant. $\square$

Consider the following relation: write $p \sim q$ if p and q belong to a connected subset of X. Proposition 3.6 shows that $\sim$ is an equivalence relation.

Definition 3.7. The equivalence classes of this equivalence relation are called the *connected components of* X.

Proposition 3.8. *Connected components of X are in fact connected and closed. Each connected set is contained in a connected component. The components are either equal or disjoint, and fill out X.*

Proof. The fact the the components fill out X follows from the observation that they are equivalence classes of an equivalence relation.

For $x \in X$ the component of x is the union of all connected sets containing x, and therefore connected by Proposition 3.6. This also implies that a connected sets is contained in a component. Connected components are closed according to Lemma 3.9. $\square$

Lemma 3.9. *If $A \subseteq X$ is a connected subset, $A \subseteq B \subseteq \overline{A}$ arbitrary, then B is connected as well.*

Proof. The crucial observations linking the connectedness of B to that of A is the following: for an open set $U \subseteq X$, $U \cap A \neq \emptyset$ is equivalent to $U \cap \overline{A} \neq \emptyset$ and hence equivalent to $U \cap B \neq \emptyset$.

Suppose that there exist open subsets $U, V \subseteq X$ such that

$$(U \cap B) \cup (V \cap B) = B$$

and $U \cap V \cap B = \emptyset$.

But then the same holds for $A : (U \cap A) \cup (V \cap A) = A$ and $U \cap V \cap A = \emptyset$. As A is connected, one of the sets $U \cap A$ or $V \cap A$ must be empty; say $U \cap A = \emptyset$. This is equivalent to $A \subseteq X - U$, which implies $\overline{A} \subseteq X - U$, hence $B \subseteq X - U$. But then $U \cap B = \emptyset$, and so B is connected. $\square$

Example 3.10 (Connected components are not necessarily open)**.** Consider $\mathbb{Q} \subseteq \mathbb{R}$ with the subspace topology. Then the only connected subsets of $\mathbb{Q}$ are the one-element sets. The connected components are one-element sets, hence closed but not open. To show this, let $A \subseteq \mathbb{Q}$ be an arbitrary subset with at least two points $p < q$. Pick an irrational number $p < \alpha < q$. Then

$$A = (A \cap \{t < \alpha\}) \cup (A \cap \{\alpha < t\})$$

provides a separation of A into two disjoint, non-empty open sets. This argument shows that no subset with more than one element in $\mathbb{Q}$ is connected.

To see that one-point sets in $\mathbb{Q}$ are not open, observe that any non-empty open set in $\mathbb{R}$ contains an open interval, hence has infinitely many rational numbers; thus any non-empty open set in $\mathbb{Q}$ must be infinite.

Definition 3.11. A topological space X is *totally disconnected* if the only connected subsets of X are one-element sets.

So far we have seen many disconnected spaces, yet we owe ourselves proving that those spaces of which we feel that are connected, are indeed so.

Theorem 3.12. *The subset* $[0,1] \subseteq \mathbb{R}$ *is connected.*

Proof. Suppose that $[0,1]$ is disconnected, that is, there exist open subsets $A, B \subseteq \mathbb{R}$ such that $A \cap B \cap [0,1] = \emptyset$, $(A \cap [0,1]) \cup (B \cap [0,1]) = [0,1]$, and both $A \cap [0,1]$ and $B \cap [0,1]$ are nonempty.

Without loss of generality we can assume that there exists $a \in A$ and $b \in B$ with $a < b$. Therefore we can write

$$[a,b] \;=\; (A \cap [a,b]) \cup (B \cap [a,b])$$

as the union of two non-empty open subsets.

The question is, where could $c \stackrel{\text{def}}{=} \sup(A \cap [a,b])$ belong?

Suppose first that $c \in A \cap [a,b]$. Then $c \neq b$, hence either $c = a$, or $a < c < b$. Since $A \cap [a,b] \subseteq [a,b]$ is open, in any case there exists a half-open interval $[c, c+\epsilon) \subseteq A \cap [a,b]$. But this means that $c < c + \frac{\epsilon}{2} \in A \cap [a,b]$, hence c cannot be a lower bound of $A \cap [a,b]$, therefore $c \neq \sup(A \cap [a,b])$.

Hence we are bound to suppose that $c \in B \cap [a,b]$. Analogously to the previous case, since $c \neq a$ and $B \cap [a,b] \subseteq [a,b]$ is open, we can find a half-open interval $(c - \epsilon, c] \subseteq B \cap [a,b]$. This however means that $c - \frac{\epsilon}{2} < c$ is also an upper bound of $A \cap [a,b]$, so c cannot be the least upper bound of $A \cap [a,b]$, a contradiction.

We have established that c cannot belong to any of A and B, hence our assumption that $[0,1]$ is disconnected was false. $\square$

Connectedness has many important consequences, the following is one of the most important.

Theorem 3.13 (Intermediate value theorem)**.** *Let X be a connected topological space, $f : X \to \mathbb{R}$ a map. If $a, b \in X$, $f(a) < \gamma < f(b)$ for some $\gamma \in \mathbb{R}$, then there exists $c \in X$ for which $f(c) = \gamma$.*

Proof. Let $A \stackrel{\text{def}}{=} f(X) \cap (-\infty, \gamma)$ and $B \stackrel{\text{def}}{=} f(X) \cap (\gamma, +\infty)$. Then $A \cap B = \emptyset$, both A and B are non-empty and open in $f(X)$. Suppose that there does not exist $c \in X$ with $f(c) = \gamma$. Then $f(X) = A \cup B$ is the disjoint union of two non-empty open subsets, hence disconnected. This however contradicts the fact that $f(X)$ being the continuous image of a connected space is itself connected. □

The connectedness notion we have introduced is not the only one imaginable. Intuition provides an alternate viewpoint: we would like to call a topological space connected if we can 'go' from any of its points to any other point. This version is made precise in the definition below.

Definition 3.14.) Let (X, τ) be a topological space, $x, y \in X$ arbitrary (not necessarily different) points. A *path in X from x to y* is a map $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

The topological space X is called *path-connected* or *arcwise connected* if for every pair of points x, y in X there exists a path γ in X joining x to y.

Remark 3.15. A path connected topological space is connected; this can be seen as follows: suppose $X = A \cup B$ is a separation of X, let $a \in A$,$b \in B$ and $f : [0,1] \to X$ a path from a to b. The image $f([0,1])$ is connected, hence must lie either completely in A or completely in B. But this contradicts the choice of a and b.

A connected topological space need not be path-connected, but it is not easy to construct such examples.

Example 3.16. Let $X = [0,1] \times [0,1]$, make it into a partially ordered set with the help of the lexicographic order. Then X with the order topology is connected, but not path-connected.

Example 3.17 (Topologists' sine curve)**.** Consider the subset

$$S \overset{\text{def}}{=} \left\{(x, \sin\tfrac{1}{x}) \,|\, 0 < x \le 1\right\} \subseteq \mathbb{R}^2 ,$$

which is the graph of the function $x \mapsto \sin\dfrac{1}{x}$ over the half-open interval $(0,1]$. As we will see shortly, the closure of S in $\mathbb{R}^2$ equals

$$\overline{S} = S \cup \{0\} \times [-1,1] .$$

We will show that $\overline{S} \subseteq \mathbb{R}^2$ with the subspace topology is connected, but not path-connected.

Proposition. *With notation as above,*

(1) $\overline{S} = S \cup \{0\} \times [-1,1]$,
(2) $\overline{S}$ *is connected,*
(3) $\overline{S}$ *is not path-connected.*

Proof. The connectedness of $\overline{S}$ is simple: the subset $S \subseteq \mathbb{R}^2$ is connected as the image of the connected set $(0,1]$; but then so is $\overline{S}$, being the closure of a connected subset of $\mathbb{R}^2$. Proof of the other two claims is to come. □

Exercise 3.18.

(1) Show that no two of $(0,1)$, $(0,1]$, *and* $[0,1]$ *are homeomorphic.*
(2) Prove that $\mathbb{R}^n \not\approx \mathbb{R}$ *whenever* $n > 1$.

Exercise 3.19. *Show that every continuous map* $f : [0,1] \to [0,1]$ *has a fixed point (i.e. there exists* $c \in [0,1]$ *such that* $f(x) = x$*). Give an example to illustrate that the same does not hold for* $[0,1)$.

Definition 3.20. Let X be a topological space. The equivalence classes of the relation between points of X given by "there is a path from x to y" are called *path* or *arc* components of X.

Theorem 3.21. *The path components of a topological space* X *are disjoint path-connected subspaces whose union is* X. *Each path-connected subspace intersects exactly one path component.*

Proof. About the same as for connectedness. □

Definition 3.22. A topological space (X, τ) is *locally connected at a point* $x \in X$, if x has a neighbourhood basis consisting of connected subsets. The space X is called *locally connected* if it is locally connected at every $x \in X$.

The topological space X is *locally path connected at a point* $x \in X$, if x has a neighbourhood basis consisting of path-connected subsets; X is *locally path-connected* if it is locally connected at every $x \in X$.

Proposition 3.23. X *is locally connected if and only if for every open subset* $U \subseteq X$, *each component of* U *is open in* X.

Proof. Let $U \subseteq X$ be an open set, $C \subseteq U$ a connected component of U. Pick a point $x \in C$, and choose a connected neighbourhood V of x such that $V \subseteq U$. Now C is connected hence $V \subseteq C$, and so $C \subseteq X$ is open.

For the other direction, assume that the components of open sets in X are themselves open. Fix $x \in X$, an open neighbourhood U of x, and let C be the connected component of U containing x. Since C is connected and open in X, X is locally connected at x. □

Exercise 3.24. *Show that a topological space* X *is locally path-connected if and only if for every open subset* $U \subseteq X$, *each path-component of* U *is open in* X.

Theorem 3.25. *Let* X *be a topological space. Each path component of* X *lies in a connected component of* X. *If* X *is locally path connected then connected components and path components coincide.*

Proof. Let $C \subseteq X$ be a connected component of X, $x \in C$ arbitrary. Let P be the path component of X containing x. Then P is connected, hence $P \subseteq C$. Suppose that $P \neq C$.

We will denote the union of all path components of X that are different from P but intersect C by Q. Each of these path components lies in C, so $C = P \cup Q$. As

X is locally path connected, the path components are open in X. But this means that $P \cup Q$ is a separation of C, a contradiction. □

Definition 3.26 (Disjoint union/topological sum)**.** Let X, Y be topological space. Then the disjoint union $X \coprod Y$ of X and Y is defined as follows. As a set it is equal to $(X \times \{0\}) \cup (Y \times \{1\})$, we equip this set with the topology making both $X \times \{0\}$ and $Y \times \{1\}$ clopen, and the inclusions

$$x \mapsto (x,0) \qquad X \to X \coprod Y$$
$$y \mapsto (y,1) \qquad Y \to X \coprod Y$$

homeomorphisms onto their images.

The disjoint union of an arbitrary collection of topological spaces can be defined analogously.

Exercise 3.27. *Check that the disjoint union of two topological spaces is a well-defined topological space.*

Exercise 3.28. *The relation "$p \ominus q$ if for every discrete valued map d on X, $d(p) = d(q)$" is an equivalence relation, the equivalence classes of which are called* quasi-components*.*
(i) Show that quasi-components are either equal or disjoint, and fill out X.
(ii) The quasi-components of a topological space are closed; each connected set is contained in a quasi-component.
(iii) Let $X \stackrel{def}{=} \{ \{ (0,0) \}, \{ (0,1) \} \} \cup \bigcup_{n=1}^{\infty} \{ \frac{1}{n} \} \times [0,1] \subseteq \mathbb{R}^2$. Then the points $(0,0)$ and $(0,1)$ are components, but not quasi-components.

Exercise 3.29. *Let $(X,\tau), (X,\sigma)$ two topologies on the same set, assume that $\sigma \subseteq \tau$. Does connectivity of a subset with respect to one topology imply anything for connectivity in the other?*

Exercise 3.30. *Let C_n be an infinite sequence of connected subspaces of a topological space X, such that for every n, one has $C_n \cap C_{n+1} \neq \emptyset$. Show that $\bigcup_{n=1}^{\infty} C_n$ is connected as well.*

Exercise 3.31. *An infinite set is always connected in the finite complement topology.*

Exercise 3.32. *Show that a discrete topological space is totally disconnected. Is the converse true?*

Exercise 3.33. *Let $A \subseteq X$ be an arbitrary subspace, $C \subseteq X$ connected. Prove that $A \cap C \neq \emptyset$ and $(X - A) \cap C \neq \emptyset$ together imply $\partial A \neq \emptyset$.*

Exercise 3.34. *If $X \subseteq \mathbb{R}^n$ is a convex subset, then X is connected.*

Exercise 3.35. *Prove that a connected metric space is either uncountable or has at most one point.*

Exercise 3.36. *Show that a connected open set in a locally path connected space is path connected.*

4. Separation axioms and the Hausdorff property

General topological spaces have little in common, since the basic axioms hardly say anything. Therefore, and also to make topological spaces resemble real life to some extent, it seems reasonable to put further restrictions on them.

Definition 4.1. A topological space X is a T_0-*space* if for every two points $x, y \in X$ there exists an open subset U with either $x \in U, y \notin U$ or $x \notin U, y \in U$.

Definition 4.2. X is called a T_1-*space*, if for every pair of points $x, y \in X$ there exists an open set U with $x \in U$ and $y \notin U$.

Note that a T_1-space is certainly a T_0-space, but the two properties differ. The T_0 property is equivalent to requiring, that the points of X can be distinguished by the collections of open sets they lie in. As one can check quickly, the T_1 property is the same as insisting that all one-point sets in X are closed.

Definition 4.3. A topological space X is called *Hausdorff* or a T_2-*space*, if for every pair of points $x, y \in X$, there exist open sets $U, V \subseteq X$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Of all the separation axioms that we will encounter, the Hausdorff property is by far the most important. We point out that a subspace of a Hausdorff space inherits the Hausdorff property.

Definition 4.4. A topological space is called *regular* or a T_3-*space*, if for every point $x \in X$ and every closed set $F \subseteq X$ not containing x, there exist disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $F \subseteq V$.

Definition 4.5. X is called *normal* or a T_4-*space* if for every pair of disjoint closed sets $F, G \subseteq X$ there exist disjoint open sets $U, V \subseteq X$ with $F \subseteq U$ and $G \subseteq V$.

Exercise 4.6. *Give examples to show that the Hausdorff property is not implied by regularity or normality.*

Exercise 4.7. *Find an example of a topological space that is T_0 but not T_1, an example which is T_1, but not T_2, and so on.*

Proposition 4.8. *A Hausdorff topological space is regular if and only if the closed neighbourhoods of any point x form a neighbourhood basis of x.*

Proof. First assume that X is a regular Hausdorff topological space. Pick a point $x \in X$, let V be an open neighbourhood of x, and set $C \stackrel{\text{def}}{=} X - V$. By regularity, there exist disjoint open subsets $U, W \subseteq X$ such that $x \in U$ and $C \subseteq W$.

Therefore $X - W$ is closed, and $X - W \subseteq X - C = V$, hence any open neighbourhood V of X contains a closed neighbourhood $X - W$ of x.

Now assume that every point $x \in X$ has a neighbourhood basis consisting of closed subsets of X. Pick $C \subseteq X$ arbitrary closed, and $x \notin C$. Let $V = X - C$. By assumption, there exists an open set $U \subseteq X$ with $\overline{U} \subseteq V = X - C$, $x \in U$. This implies that $C \subseteq X - \overline{U}$, and $U \cap (X - \overline{U}) = \emptyset$, and so X is regular. $\square$

Corollary 4.9. *A subspace of a regular Hausdorff space is regular Hausdorff.*

Proof. Let $A \subseteq X$ be a subspace of a regular Hausdorff space. The Hausdorff propety has already been taken care of; regularity comes from the fact that by intersecting a closed neighbourhood basis of $x \in A$ in X with A, we obtain a closed neighbourhood basis of x in the subspace A. $\square$

4.1. **More on the Hausdorff property.** Here we collect some observations on the Hausdorff separation property, that are important in algebraic geometry and the theory of manifolds. In particular, we address the issue to what extent can we control the Hausdorff property upon gluing.

We have seen earlier, that by gluing together Hausdorff topological spaces (even finitely many) we can lose this important property. To get a better grasp on this notion, we present an alternate characterization, which might be at first unusual-looking, but has far-reaching consequences. Let X be a topological space. The key tool is the so-called *diagonal map* $\Delta_X : X \to X \times X$ given by $x \to (x, x)$.

Exercise 4.10. *Construct the diagonal map using the universal property of the product topology. Show that $\Delta_X : X \to X \times X$ is indeed continuous, even better, it is a homeomorphism onto its image.*

Proposition 4.11. *A topological space X is Hausdorff if and only if $\Delta_X(X) \subseteq X \times X$ is a closed subset.*

In other words, X is Hausdorff if and only if Δ_X is a homemorphism onto a closed subset of $X \times X$.

Proof. The subset $\Delta_X(X) \subseteq X \times X$ is closed if and only if $U \stackrel{\text{def}}{=} X \times X - \Delta_X(X) \subseteq X \times X$ is open. A pont $(x, x') \in X \times X$ is in U if and only if $x \neq x'$. By definition of the product topology, U is open if and only if for every point $(x, x') \in U$ there exist open sets $V, W \subseteq X$ such that $(x, x') \in V \times W$, and $V \times W \subseteq U = X \times X - \Delta_X(X)$. This latter property translates into $(V \times W) \cap \Delta_X(X) = \emptyset$. Equivalently, we require that $(x, x') \in V \times W$ and $V \cap W = \Delta_X^{-1}(V \times W) = \emptyset$, which is the same as asking that $x \in V$, $x' \in W$, and $V \cap W = \emptyset$. $\square$

Theorem 4.12. *Let Y be a Hausdorff, X an arbitrary topological space, $f, g : X \to Y$ continuous maps. If $f|_S = g|_S$ for a dense subset $S \subseteq X$, then $f = g$.*

The example with the real line with the origin doubled illustrates that the Hausdorff property is crucial.

Proof. We consider the product map $f \times g : X \to Y \times Y$. The subset of X on which f and g agree is equal to $(f \times g)^{-1}(\Delta_Y(Y))$. Since Y is Hausdorff, this set is closed, on the other hand it contains the dense subset S. Therefore $(f \times g)^{-1}(\Delta_Y(Y)) = X$, and so $f = g$. □

One of the most important applications of Proposition 4.11 is a criterion for the Hausdorffness of a topological space glued together from open pieces.

Theorem 4.13. *Let (X, τ) be the topological space obtained by gluing together the collection $\{(X_\alpha, \tau_\alpha) \,|\, \alpha \in I\}$. Then X is Hausdorff exactly if each X_α is Hausdorff, and $\Delta_X(X_\alpha \cap X_\beta)$ is closed in $X_\alpha \times X_\beta$ for every $\alpha, \beta \in I$.*

Proof. Since $\{X_\alpha\}$ forms an open cover, the open subsets $X_\alpha \times X_\beta$ cover the product space $X \times X$. Recall that closedness is a local property, hence a subset $A \subseteq X \times X$ is closed if and only if $A \cap X_\alpha \cap X_\beta \subseteq X_\alpha \cap X_\beta$ is closed.

Let us identify X and its image $\Delta_X(X)$ via the diagonal map (which is a homeomorphism). Under this identifications $\Delta_X(X) \cap (X_\alpha \times X_\beta)$ corresponds to $X_\alpha \cap X_\beta \subseteq X$. Therefore, Δ_X has closed image in $X \times X$ if and only if the restriction of the diagonal map $X_\alpha \cap X_\beta \to X_a \times X_\beta$ has closed image for every $\alpha, \beta \in I$.

If $\alpha = \beta$, then this just says that X_α is closed in $X_\alpha \times X_\alpha$ (via the diagonal map), that is, X_α is Hausdorff. For $\alpha \neq \beta$ we obtain the condition in the Theorem. □

Exercise 4.14. *Check what happens for the real line with the origin doubled.*

Exercise 4.15. *Show that in a Hausdorff topological space a sequence can have at most one limit.*

Exercise 4.16. *Prove that a subspace of a Hausdorff topological space is itself Hausdorff with respect to the subspace topology.*

Exercise 4.17. *Decide whether the product of two Hausdorff spaces is Hausdorff.*

5. Compactness and its relatives

The notion of a compact space is a vast generalization of closed bounded sets in Euclidean spaces. As we will see, it has far-reaching consequences, among others, a real-valued function on a compact topological space takes on its extremal values.

As it turns out, even weak versions of compactness like local compactness or paracompactness prove to be fundamental for much of geometry.

Definition 5.1. A *covering* or *cover* of $\mathcal{C}$ of a topological space X is a collection of subsets of X whose union is X. A covering $\mathcal{C}$ is called *open*, if all of its elements are open subsets of X. A subcover of a covering $\mathcal{C}$ is a subset of $\mathcal{C}$ such that the union of its elements is still X.

Definition 5.2 (Compactness)**.** A topological space X is *compact*, if every open cover of X has a finite subcover.

This is called the Heine–Borel property. Now we will work on a useful characterization in terms of closed subsets.

Definition 5.3 (Finite intersection property)**.** Let $\mathcal{C}$ be an arbitrary collection of subsets of X. We say that $\mathcal{C}$ has the *finite intersection property* or FIP for short, if the intersection of any finite subcollection of $\mathcal{C}$ is non-empty.

Proposition 5.4. *A topological space X is compact if and only if for every collection $\mathcal{C}$ of closed subsets of X with the finite intersection property the intersection of all sets in $\mathcal{C}$ is non-empty.*

Proof. Let us assume that X is compact, and let

$$\mathcal{C} \stackrel{\text{def}}{=} \{C_\alpha \,|\, \alpha \in I\}$$

a collection of closed subsets of X with the finite intersection property, suppose that

$$\bigcap_{\alpha\in I} C_\alpha \;=\; \emptyset \;.$$

Consider

$$\mathcal{U} \stackrel{\text{def}}{=} \{X - C_\alpha \,|\, \alpha \in I\} \;\;,$$

the collection of the complements of the elements of $\mathcal{C}$. Then

$$X \;=\; X - \emptyset \;=\; X - \left(\bigcap_{\alpha\in I} C_\alpha\right) \;=\; \bigcup_{\alpha\in I} (X - C_\alpha) \;,$$

that is, $\mathcal{U}$ is an open cover of X. By compactness, $\mathcal{U}$ has a finite subcover $X - C_{\alpha_1}, \dots, X - C_{\alpha_k}$. This means that

$$X \;=\; \bigcup_{i=1}^{k}(X - C_{\alpha_i}) = X - \bigcap_{i=1}^{k} C_{\alpha_i} \;.$$

Therefore $\bigcap_{i=1}^{k} C_{\alpha_i} = \emptyset$, which contradicts the finite intersection property of $\mathcal{C}$.

The other direction is completely analogous. $\square$

Theorem 5.5. *Any compact subspace of a Hausdorff topological space is closed.*

Proof. Let X be a T_2-space, $C \subseteq X$ a compact subset, $x \in X - C$. For $a \in C$ let U_a, V_a be disjoint open subsets of X such that $a \in U_a$ and $x \in V_a$. Note that

$$\bigcup_{a\in C} (U_a \cap C) \;=\; C$$

is an open cover of C, hence by compactness we can find finitely many points $a_1, \dots, a_k$ for which

$$\bigcup_{i=1}^{k} (U_a \cap C) \;=\; C \;.$$

Let $U(x) \stackrel{\text{def}}{=} U_{a_1} \cup \cdots \cup U_{a_k}$, and $V(x) \stackrel{\text{def}}{=} V_{a_1} \cap \cdots \cap V_{a_k}$. Then $U(x), V(x) \subseteq X$ are open sets, $U(x) \supseteq C$, and $V(x) \cap C = \emptyset$. Therefore

$$x \in V(x) \subseteq X - U(x) \subseteq X - C .$$

Since this holds for any $x \notin C$, we have

$$X - C = \bigcup_{x \notin C} V(x) ,$$

which means that $X - C$ is open, hence $C \subseteq X$ is closed. □

Proposition 5.6. *The image of a compact topological space under a continuous map is compact.*

Proof. Look at the inverse images of an open cover of the target. □

Proposition 5.7. *Let X be a compact topological space, $A \subseteq X$ a closed subset. Then A with the subspace topology inherited from X is compact as well.*

Proof. Let $\mathcal{C} = \{V_\alpha \mid \alpha \in I\}$ an arbitrary open cover of A. By the definition of the subspace topology, there exists a collection $\mathcal{C}' = \{U_\alpha \mid \alpha \in I\}$ of open sets in X such that $U_\alpha \cap A = V_\alpha$; as $\mathcal{C}$ covers A, so will $\mathcal{C}'$.

Consider the open cover $\{X - A\} \cup \mathcal{C}'$ of X. Since X is compact, $\{X - A\} \cup \mathcal{C}'$ has a finite subcover $U_1, \ldots U_k$. If $X - A$ shows up in this finite collection, discard it. The remaining ones will all belong to $\mathcal{C}'$, and will still cover A (as $X - A$ is disjoint from A) hence their intersection with A will produce a finite subcover of $\mathcal{C}$. □

We have explicitly pointed out earlier, that a continuous bijection between topological spaces is in general not a homemorphism. Here is one common situation, however, when it is. This result will prove to be very useful in many contexts.

Proposition 5.8. *Let X be a compact topological space, Y Hausdorff, $f : X \to Y$ a continuous bijection. Then f is a homemorphism.*

Proof. What we need to show is that $f^{-1} : Y \to X$ (which is a function now since f is bijective) is a continuous function. This is equivalent to requiring that f be a closed map.

Let $A \subseteq X$ be a closed subset. Since X is compact, A is compact as well. The image $f(A) \subseteq Y$ is also compact; a compact subset of a Hausdorff topological space is closed, therefore we are done. □

Roughly speaking, compact topological spaces have 'few' open sets, while Hausdorff ones have 'many' open sets. A not very convincing argument for this rule of thumb is to look at the two extremes: a trivial topological space is always compact (but never Hausdorff unless X is empty), a discrete topological space is always Hausdorff, but only compact, if X has finitely many elements. In this sense, compact Hausdorff spaces represent a happy middle ground. Another way to look at this idea is the following.

Exercise 5.9. *Let X be a set, τ_1, τ_2 two topologies on X such that (X, τ_1) and (X, τ_2) are both compact Hausdorff spaces. Then $\tau_1 \nsubseteq \tau_2$ and $\tau_2 \nsubseteq \tau_1$.*

Theorem 5.10. *$I \overset{def}{=} [0,1] \subseteq \mathbb{R}$ is compact.*

Proof. Let $\mathcal{U}$ be an open cover of I, set

$$S \overset{\text{def}}{=} \{\alpha \in I \mid [0,\alpha] \text{ is covered by a finite subcollection of } \mathcal{U}\} \ ,$$

and let $\beta \overset{\text{def}}{=} \sup S$. Observe that S must be an interval $[0,\beta)$ or $[0,\beta]$.

Assume that $S = [0,\beta)$, take $U \in \mathcal{U}$ with $\beta \in \mathcal{U}$. As an open set in $\mathbb{R}$ is a union of finite open intervals, U must contain an interval of the form $[\alpha,\beta]$. But then we can consider the hypothetical finite cover of $[0,\alpha]$ and U to obtain a finite cover of $[0,\beta]$ with elements of U. Therefore $S = [0,\beta]$.

A very similar argument shows that β must be equal to 1. □

As any two finite closed intervals are homeomorphic to each other (you can find a linear function doing the job), $[a,b]$ is compact for every $a, b \in \mathbb{R}$.

Corollary 5.11. *A subset $X \subseteq \mathbb{R}$ is compact if exactly if it is closed and bounded.*

Proof. Assume first that X is compact. Then it is also closed since it $\mathbb{R}$ is Hausdorff in the Euclidean topology. For boundedness, consider the open cover

$$\{(-n,n) \mid n \in \mathbb{N}\} \ .$$

To go in the other direction, take a subset $X \subseteq \mathbb{R}$, which is closed and bounded. Boundedness is equivalent to saying that $X \subseteq [-n,n]$ for some n. But then X is a closed subspace of the compact topological space $[-n,n]$, hence itself compact. □

Remark 5.12. As we will see later, closed and bounded subspaces of a metric space will typically not be compact, one needs stronger hypotheses.

Theorem 5.13 (Extremal value theorem)**.** *If $f : X \to \mathbb{R}$ is a continuous function, X compact, then f assumes a smallest and a largest value on X.*

Proof. Since X is compact, so is $f(X) \subseteq \mathbb{R}$. But then $f(X)$ is closed and bounded, hence $\sup f(X) < \infty$, and so $\sup f(X) \in f(X)$. Same argument holds for the infimum. □

The following closely related notions mimic the Bolzano–Weierstrass version of compactness.

Definition 5.14. A topological space X is *limit point compact* if every infinite subset of X has an a limit point. X is said to be *sequentially compact* if every sequence of points in X has a convergent subsequence.

One has to watch out, as in a general topological space the non-uniqueness of limits of sequences leads to the fact the sequential compactness is fairly different from compactness.

Proposition 5.15. *A compact topological space X is limit point compact.*

Proof. Let $A \subseteq X$ be an infinite subset of the compact space X. Suppose for a contradiction that A has no limit point. Then A contains all of its limit points, hence A is closed.

For an arbitrary $a \in A$ an open subset $a \in U_a \subset X$ such that $U_a \cap A = X$. Then

$$X = (X - A) \cup \bigcup_{a \in A} U_a$$

is an open cover of X. Since X is compact, the above cover has a finite subcover. Only finitely many of the U_a's can show up in the finite subcover. As $(X - A) \cap A\emptyset$, all the points of A are contained in the finitely many U_a's. But every one of them contains exactly one common point with A. Therefore A is finite. □

Proposition 5.16. *Let $f : Z \to W$ be a closed perfect map. Then f is also proper.*

For the proof we will need the following lemma.

Lemma 5.17. *If $f : X \to Y$ is a closed map, then for every $y \in Y$ and every open set $f^{-1}(y) \subseteq U \subseteq X$ there exists an open set $W \subseteq Y$ with $y \in W$ and $f^{-1}(W) \subseteq U$.*

Proof. Let $y \in Y$ and $f^{-1}(y) \subseteq U \subseteq X$ be arbitrary. Then $X - U \subseteq X$ is closed, hence $f(X - U) \subseteq Y$ is closed as well, f being a closed map. Moreover $f(X - U) \cap \{y\} = \emptyset$.

Set $W \stackrel{\text{def}}{=} Y - f(X - U) \subseteq Y$. Then W is an open subset of Y containing y. We are left with showing that $f^{-1}(W) \subseteq U$: if $w \in f^{-1}(W)$, then $f(w) \in W$, hence $f(w) \notin f(X - U)$, and so $w \in U$ (since $w \notin U$ implies $w \in X - U$ hence $f(w) \in f(X - U)$, a contradiction). □

Proof of Proposition 5.16. We will prove that Y compact implies X compact, the general case follows by restriction.

Consider an arbitrary open cover $\mathcal{U}$ of X. The plan is to somehow cook up an open cover of Y using $\mathcal{U}$, and use the compactness of Y to find a finite subcover of $\mathcal{U}$ Here is the construction of an open cover of Y: as f is perfect, $f^{-1}(y) \subseteq X$ is compact for every $y \in Y$. Therefore there exist finitely many elements $U_{y,1}, \ldots U_{y,k_y}$ of $\mathcal{U}$ covering $f^{-1}(y)$. Consider the open set $U_y \stackrel{\text{def}}{=} U_{y,1} \cup \cdots \cup U_{y,k_y} \supseteq f^{-1}(y)$. By the Lemma there exists an open set $y \in W_y \subseteq Y$ such that $f^{-1}(W_y) \subseteq U_y$. Then $Y - f(X)$, and the W_y's form an open cover of the compact topological space Y. Therefore there exists a finite subcover; the corresponding U_y's form a finite subcover of $\mathcal{U}$. □

Proposition 5.18. *If X is a compact topological space, then $\pi_Y : X \times Y \to Y$ is a closed map.*

Proof. Let $y \in Y - \pi_Y(C)$, that is, let y be an element of Y such that for every $x \in X$, the pair $(x, y) \notin C$.

Then for every $x \in X$ there exists open sets $U_x \subseteq X$ and $V_x \subseteq Y$ such that $x \in U_x$ and $y \in V_x$ and $(U_x \times V_x) \cap C = \emptyset$.

Since X is compact, there exist finitely many points $x_1, \ldots, x_k \in X$ such that $U_{x_1} \cup \cdots \cup U_{x_k} = X$. Let $V \stackrel{\text{def}}{=} V_{x_1} \cap \cdots \cap V_{x_k}$. Then

$$(X \times V) \cap C \;=\; (U_{x_1}] \cup \cdots \cup U_{x_k}) \times ((V_{x_1} \cap \cdots \cap V_{x_k}) \cap C) \;=\; \emptyset\ .$$

Therefore $y \in V \subseteq Y - \pi_Y(C)$, $V \subseteq Y$ open. Hence $Y - \pi_Y(C) \subseteq Y$ is open, and so $\pi_Y(C) \subseteq Y$ is closed. □

Corollary 5.19. *Let X,Y be topological spaces, X compact. Then $\pi : X \times Y \to Y$ is proper.*

Proof. □

Definition 5.20. A map $f : X \to Y$ between topological space is called *proper* if the inverse image of every compact subset of Y is compact in X. The map f is called *perfect* if the inverse image of every one-point set is compact.

Proposition 5.21. *If X is compact, then $\pi_Y : X \times Y \to Y$ is proper.*

Proof. As π_Y is a closed map according to Proposition 5.18, Proposition 5.16 implies that it is proper once it is perfect. But the preimage of any one-point set in Y is X, which is compact. Therefore π_Y is proper. □

Corollary 5.22. *The product of two compact topological spaces is compact.*

Proof. Consider the perfect map $\pi_Y : X \times Y \to Y$. Since Y is compact, so is $\pi_Y^{-1}(Y) = X \times Y$. □

Corollary 5.23. *If X_1,..., X_k are compact topological spaces, then $X_1 \times \cdots \times X_k$ is compact as well.*

Proof. Use induction on the number of factors. □

Remark 5.24. A surprising non-trivial result known as Tychonoff's theorem says the an *arbitrary* product of compact topological spaces is compact. The proof requires advanced tools from set theory.

Corollary 5.25. *The n-dimensional cube $[0,1]^n \subseteq \mathbb{R}^n$ is compact.*

Corollary 5.26. *A subset $X \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. Assume first that X is compact. Then it is automatically closed. Cover X by intersections with $\mathcal{B}(0,k)$, where k runs through all natural numbers. Compactness implies that finitely many of them covers X, as they are nested, the largest one contains X, hence X is bounded as well.

For the other direction, let $X \subseteq \mathbb{R}^n$ be a closed and bounded subset. Then $X \subseteq \mathcal{B}(0,k)$ for some positive k, which is in turn contained in $[-k,k]^n$, a compact set. Then X being a closed subset of a compact set is itself compact. □

Exercise 5.27 (Tube lemma)**.** *Let X be an arbitrary, Y a compact topological space, $x_0 \in X$ an arbitrary point, $N \subseteq X \times Y$ an open subset containing $\{ x_0 \} \times Y$. Prove that there exists an open neighbourhood of W of x_0 in X such that $N \supseteq W \times Y$.*

Exercise 5.28. *Take two disjoints compact subspaces $F, G \subseteq X$, where X is Hausdorff. Prove that there exist disjoint open sets $U, V \subseteq X$ for which $F \subseteq U$ and $G \subseteq V$.*

Exercise 5.29. *Let X be a non-empty compact Hausdorff space with no isolated points. Show that X must be uncountable.*

5.1. **Local compactness and paracompactness.** Here we treat two very useful weakenings of compactness. As it turns out, one finds many more spaces in practice that are locally compact or paracompact, than that are actually compact. More often than not, local compactness and paracompactness are discussed in the presence of the Hausdorff property.

Definition 5.30 (Local compactness)**.** Let X be a topological space, $x \in X$. We say that X is *locally compact at x*, if there exists a compact subset $C \subseteq X$ which contains a neighbourhood of x. The space X is called *locally compact* if it is locally compact at every one of its points.

In other words, a topological space is compact precisely if every point has a compact neighbourhood. Under these circumstances, every point has an open neighbourhood, whose closure is compact.

Exercise 5.31. *Prove that $\mathbb{R}^n$ is locally compact, but $\mathbb{Q}$ is not.*

Proposition 5.32. *Let X be a locally compact Hausdorff space, $x \in X$ an arbitrary point. Then each neighbourhood of x contains a compact neighbourhood of x.*

Proof. Let $x \in C \subseteq X$ be a compact neighbourhood of x, $U \subseteq X$ an arbitrary neighbourhood of x, which without loss of generality we can take to be open in X. Let $V \subseteq C \cap U$ be an open set in X (since C is a neighbourhood of x in X, it contains an open neigbourhood of x, take such a set and intersect it with U). Then $\overline{V} \subseteq C$ is a compact Hausdorff space, hence it is regular as well. Therefore, there exists a neighbourhood $N \subseteq V$ of x in C, which is closed in $\overline{x}$, whence closed in X as well. Since N is closed in the compact topological space C, it is itself compact. The subspace N is a neighbourhood of x in $\overline{V}$, moreover, as $N = N \cap V$, it is a neighbourhood of x in the open set V, and so in X as well. □

Given that compact topological spaces have many desirable properties, it is important to know how more general spaces can be embedded as hopefully 'large' parts of a compact space. Thus, one might need a procedure that produces from a topological space X a compact space by adding as little 'extra stuff' as necessary.

The least amount of extra stuff one can imagine is of course one point. Although it might seem dubious that we can indeed produce a compact space by adding a single point, when properly done, it works under fairly general circumstances. This is one place where local compactness comes in handy.

Definition 5.33 (One-point compactification)**.** Let (X,τ) be a locally compact Hausdorff topological space, and set

$$X^+ \overset{\text{def}}{=} X \cup \{\infty\} \ ,$$

where the symbol ∞ stands for an arbitrary point *not* in X.

We define a topology τ^+ on X^+ in the following way. $U \subseteq X^+$ is *open with respect to* τ^+ if either $U \subseteq X$ and $U \in \tau$ (that is, U is an open subset of X), or if $U = X^+ - C$, where $C \subseteq X$ is compact.

The space (X^+, τ^+) is called the *one-point compactification of* X.

Theorem 5.34. *With notation as above, (X^+, τ^+) is a compact Hausdorff topological space. The topology τ^+ is the only one making the set X^+ into a compact Hausdorff space with $\tau^+|_X = \tau$.*

Proof. First of all, note that $\emptyset \in \tau$ implies $\emptyset \in \tau^+$, and $X^+ \in \tau^+$ since $\emptyset \subseteq X$ is compact.

Next we check that the intersection of two open sets in X^+ is open as well. Let $U, V \subseteq X^+$ be open sets. According to the definition of τ^+, there are three cases, depending on how many of the two come from open subsets of X. If both, then $U \cap V \in \tau \subseteq \tau^+$, and we are done. If none, then we are again fine, since the union of two compact subsets of X is again compact. Let us now deal with the case when $U \subseteq X$ is open, and $V \subseteq X^+$ has compact complement C. Then $U \cap V = U - C$, which is open in X, as $C \subseteq X$ is closed; note that here we make use of the fact that X is Hausdorff.

Let now $\{U_\alpha \,|\, \alpha \in I\}$ be an arbitrary collection of open subsets of X^+. If all the U_α's are open subsets of X, then their union is certainly open. Assume that there exists $\beta \in I$ whose complement $C_\beta \subseteq X$ is compact. Then

$$X^+ - U \;=\; \bigcap_{\alpha \in I} (X^+ - U_\alpha) \;=\; C \cap \left(\bigcap_{\alpha \in I\ ,\ \alpha \neq \beta} (X - U_\alpha) \right)$$

is a closed subspace of C, hence compact. Therefore arbitrary unions of open sets in X^+ are open, and so τ^+ is indeed a topology.

The next step if to prove that (X^+, τ^+) is compact. Let $\{U_\alpha \,|\, \alpha \in I\}$ again be an open cover of X^+, then there exists $\beta \in I$ for which $\infty \in U_\beta$. By definition $X^+ - U_\beta$ is compact, and hence covered by a finite subcollection of the other open sets U_α, hence X^+ is compact.

For the Hausdorff property, it is enough to separate ∞ from points of X. To this end, let $x \in X$ be an arbitrary point. Since X is locally compact, we can find a neighbourhood $U \subseteq X$ of x (relative to τ) such that $\overline{U} \subseteq X$ is compact. But then

U, and the open neighbourhood $X^+ - \overline{U}$ of $\infty \in X^+$ are disjoint open sets giving a separation of x and ∞.

We will now prove that τ^+ is the only topology on X^+ with the required properties. To this end, choose any topology σ on X^+ such that (X^+, σ) is a compact Hausdorff topological space with $\sigma|_X = \tau$. Let $U \in \sigma$ be any open set. Then its complement $C \stackrel{\text{def}}{=} X - U \subseteq X^+$ is closed, hence compact. If $C \subseteq X$, then $U \in \tau^+$ by definition. If $C \nsubseteq X$, then $U \subseteq X$, moreover $U \in \tau$, so $U \in \tau^+$ as well. This means that $\sigma \subseteq \tau^+$.

For the other containment between the topologies, take an open set $U \in \tau$. As $X \subseteq X^+$ has the subspace topology, there exists an open set $U' \in \sigma$ such that $U' \cap X = U$. Since points are closed in a Hausdorff space, $X \subseteq X^+$ is open with respect to σ, hence $U = U' \cap X \in \sigma$. Last, let $C \subseteq X$ be a compact subset in τ, then it is also compact in σ (compactness does not depend on the space C is contained in), hence C is closed with respect to σ. But then $X^+ - C$ is open in σ. Therefore $\sigma = \tau^+$, and we are done. $\square$

In the case when X was compact to begin with, $X^+ = X \cup \{\infty\}$, where ∞ is an isolated point of X^+; both subsets X and $\{\infty\}$ are clopen in X.

Let us have a look at how continuous functions extend to one-point compactifications.

Proposition 5.35. *Let X,Y be locally compact Hausdorff spaces, $f : X \to y$ a continuous map. Then f extends to a continuous map $f^+ : X^+ \to Y^+$ by setting $f^+(\infty_X) = \infty_Y$ if and only if f is proper.*

Recall that f proper means that the inverse image of a compact subset of Y is compact in X. Before proceeding with the proof, let us unwind what the extension f^+ means. One says that f^+ is an extension of f, if $f^+ : X^+ \to Y^+$ is a continuous functions with $f^+(X) \subseteq Y$, and with the restriction $f^+|_X = f$.

Proof. Assume first that the map f is proper. Observe that independently of the conditions, $f^+ : X^+ \to Y^+$ exists as a function between sets, hence we only need to check that it is continuous. Take an open subset $U \subseteq Y^+$. If $U \subseteq Y$, then we are done since

$$(f^+)^{-1}(U) \;=\; f^{-1}(U) \subseteq X$$

is open in X, hence it is open in X^+ as well. Suppose that $U = Y^+ - C$, with $C \subseteq Y$ compact. In this case

$$(f^+)^{-1}(U) \;=\; X^+ - f^{-1}(C)$$

is open in X^+, as $f^{-1}(C) \subseteq X$ is compact by the properness of f. But X^+ is Hausdorff, hence $X^+ - f^{-1}(C) \subseteq X^+$ is closed In the other direction, if f^+ extends f with $f^+(\infty_X) = \infty_Y$, then $(f^+)^{-1}(\infty_Y) = \infty_X$, hence $(f^+)^{-1}(Y) = X$. For a compact subset $C \subseteq Y$, we have that $C \subseteq Y$ is closed, thus $(f^+)^{-1}(C) = f^{-1}(C) \subseteq X$ is closed, and so compact. This means that f is proper. $\square$

Corollary 5.36. *A proper map $f : X \to Y$ between locally compact Hausdorff spaces is closed.*

Proof. By Proposition 5.35 there exists a continuous extension $f^+ : X^+ \to Y^+$. Let $F \subseteq X$ be an arbitrary closed subset. Then $F \cup \{\infty\} \subseteq X^+$ is also closed, hence compact. Therefore $f^+(F \cup \{\infty\}) \subseteq Y^+$ being the image of a compact space is again compact, hence closed in Y^+, since Y^+ is Hausdorff. This implies that

$$f(F) = F \cup \{\infty\} \cap Y$$

is a closed subset of Y. □

Theorem 5.37. *Let X be a Hausdorff space. Then the following are equivalent.*

(1) X is locally compact.
(2) X is a locally closed subspace of a compact Hausdorff space.
(3) X is a locally closed subspace of a locally compact Hausdorff space.

Proof. As we have seen, a locally compact Hausdorff space is an open subset of its one-point compactification, which is compact and Hausdorff; therefore (1) implies (2). The implication (2) $\Rightarrow$ (3) is clear.

For the remaining statement, let $X \subseteq Y$ be a locally closed subset of the locally compact Hausdorff topological space, $F \subseteq Y$ closed, $U \subseteq Y$ open, and $X = F \cap U$. Then F is locally compact, and $X \subseteq F$ is open, hence it is locally compact as well. This proves (3) $\Rightarrow$ (1). □

We proceed to an area of topology which is of utmost importance for the construction and good properties of differentiable manifolds.

Lemma 5.38. *Let X be a second countable locally compact Hausdorff space. Then it admits a countable base of open sets with compact closures.*

Proof. Since X is second countable, it has a countable base of open sets $\{V_n\}$. For every point $x \in X$ there exists an open set $U_x \subseteq X$ whose closure is compact. The collection $\{V_n\}$ is a basis, so there is an integer $n(x)$ such that $V_{n(x)} \subseteq U_x$. The closure of $V_{n(x)}$ is a closed subspace of the compact set $\overline{U_x}$, hence itself compact. Therefore the collection of V_n's with compact closure forms a countable basis for the topology of X. □

Definition 5.39. Let $\mathcal{U}, \mathcal{V}$ be open covers of the topological space X. We say that $\mathcal{U}$ *refines* $\mathcal{V}$, if every element $U_\alpha \in \mathcal{U}$ is contained in some element $V_\beta \in \mathcal{V}$.

Naturally, a subcover of a cover is always a refinement, but the converse does not hold. It can easily happen that no element of a refinement belongs to the original cover.

Definition 5.40. An open cover $\mathcal{U}$ of X is called *locally finite*, if every point $x \in X$ has a neighbourhood, which is disjoint from all but finitely many elements of $\mathcal{U}$.

Example 5.41. The open cover $\{(n-1, n+1) \mid n \in \mathbb{N}\}$ of the real line is locally finite, but neither the open cover $\{(-\infty, a) \mid a \in \mathbb{Z}\}$, nor $\{(-n, n) \mid n \in \mathbb{N}\}$ is locally finite.

Definition 5.42. A topological space X is called *paracompact*, if every open cover $\mathcal{U}$ of X has a locally finite refinement.

Remark 5.43. Note that very often the Hausdorff property is included in the definition of paracompactness. This is due the fact that paracompactness is primarily used in the Hausdorff setting.

By checking the definition, it is immediate that the disjoint union of an arbitrary collection of compact sets is paracompact. It must be pointed out, however, that paracompactness is often hard to check, and does not behave nicely.

Exercise 5.44. *Show that a closed subset of a paracompact topological space is again paracompact.*

Remark 5.45. An open subset of a paracompact topological space is not necessarily paracompact.

A classic example of a paracompact topological space is the Euclidean space $\mathbb{R}^n$. Although this will follow from results we will prove later on, it is important enough to discuss it on its own as well.

Proposition 5.46. $\mathbb{R}^n$ *is a paracompact.*

Proof. Let $\{U_\alpha \mid \alpha \in I\}$ be an open cover of $\mathbb{R}^n$. Pick a point $x \in \mathbb{R}^n$, then there exists an open ball $\mathcal{B}(x, r_x)$ with radius $r_x < 1$ contained in some $U_{\alpha(x)}$. For every natural number N finitely many of the balls $\mathcal{B}(x, r_x)$ are enough to cover the compact set $\overline{\mathcal{B}(0,N)} - \mathcal{B}(0, N-1)$, let these be $\mathcal{B}(x_{N,1}, r_{x_{N,1}}), \ldots \mathcal{B}(x_{N,i_N}, r_{x_{N,i_N}})$. Let us write $V_{N,j}$ for the finitely many corresponding open sets.in the cover.

The collection $V_{N,j}$ (with $N \in \mathbb{N}$ and $1 \leq j \leq i_N$) cover $\mathbb{R}^n$, moreover, they refine $\{U_\alpha \mid \alpha \in I\}$ in the sense that every $V_{N,j}$ lies in some element of this cover. In addition, as we will now show, the $V_{N,j}$'s form a locally finite cover, that is, every $x \in \mathbb{R}^n$ has a neighbourhood which intersects only finitely many of the $V_{N,j}$'s. To see this, note that $V_{N,j}$ is a ball of radius ≤ 1 touching $\overline{\mathcal{B}(0,N)} - \mathcal{B}(0, N-1)$. The triangle inequality then implies that any bounded region of $\mathbb{R}^n$ meets only finitely many of the balls $V_{N,j}$.

□

Our main result regarding paracompactness is the following.

Theorem 5.47. *A second countable locally compact Hausdorff topological space is paracompact.*

Exercise 5.48. *If a topological space X is paracompact, then it is normal as well.*

The significance of paracompactness for geometry stems from the existence of partitions of unity, an absolutely fundamental tool in real differential geometry.

Definition 5.49. Let X be a topological space, $f : X \to \mathbb{R}$ an arbitrary real-valued function. The *support of* f is defined as

$$\operatorname{Supp} f \stackrel{\text{def}}{=} \overline{\{x \in X \mid f(x) \neq 0\}} \ .$$

Definition 5.50. Let $\{U_\alpha \mid \alpha \in I\}$ be an open cover of the space X; a *partition of unity* subordinate to this cover is a collection of continuous maps

$$\{g_\alpha : X \to [0,1] \mid \alpha \in I\}$$

satisfying the properties

(1)
$$\sum_{\alpha \in I} f_\alpha \ = \ 1 \ ,$$

(2) there exists a locally finite open refinement $\{V_\alpha \mid \alpha \in I\}$ such that
$$\operatorname{Supp} f_\alpha \ \subseteq \ V_\alpha$$
for each $\alpha \in I$.

Theorem 5.51. *Let X be a paracompact topological space, $\mathfrak{U}$ an arbitrary open cover of X. Then there exists a partition of unity subordinate to the cove $r\mathfrak{U}$.*

5.2. **Compactness in metric spaces.** This subsection contains a standard account of the various equivalent characterizations of compact metric spaces. As we will see, these characterizations mostly conform to our intuition trained on closed and bounded subspaces of $\mathbb{R}^n$. In this subsection (X, d) always denotes a metric space equipped with the topology induced from d.

Definition 5.52. A sequence (x_n) in X is a *Cauchy sequence*, if for every $\epsilon > 0$ there exists a natural number N_ϵ such that $d(x_n, x_m) < \epsilon$ whenever $m, n \geq N$. The metric space X is called *complete*, if every Cauchy sequence converges.

Note that it is a standard fact from multivariable calculus that $\mathbb{R}^n$ with its standard metric is a complete metric space.

Lemma 5.53. *The metric space X is complete precisely if every Cauchy sequence has a convergent subsequence.*

Proof. One direction is easy, if X is complete, then every Cauchy sequence converges, hence it trivially has a convergent subsequence (itself).

For the converse, consider a Cauchy sequence (x_n) in X with $(x_{n_k}) \to x$ being a convergent subsequence. We show that the entire sequence x_n converges to x. Fix $\epsilon > 0$ arbitrary, and pick N_ϵ to be large enough so that $d(x_n, x_m) < \epsilon/2$ whenever $n, m \geq N_\epsilon$. Such a natural number exists by virtue of the fact that (x_n) is a Cauchy sequence.

Next, choose an integer N'_ϵ for which $d(x_{n_k}, x) < \epsilon/2$ holds whenever $n_k \geq N'_\epsilon$. This we can do since the sequence x_{n_k} converges to x. If $n \geq \max N_\epsilon, N'_\epsilon$, then we have

$$d(x_n, x) \ \leq \ d(x_n, x_{n_k}) + d(x_{n_k}) \ \leq \ \epsilon \ ,$$

hence $x_n \to x$. □

Exercise 5.54. *Prove that a convergent sequence in a metric space is a Cauchy sequence. Give an example to show that the converse does not hold in general.*

To be able to characterize compact metric spaces, we need a more restrictive version of boundedness. It turns out that while perfectly adequate in $\mathbb{R}^n$, boundedness as we know it is too weak in general.

Definition 5.55. A metric space X is *totally bounded*, if for every $\epsilon > 0$ X can be covered by finitely many balls of radius ϵ.

Exercise 5.56. *Show that boundedness and total boundedness are equivalent concepts in (finite-dimensional) Euclidean spaces.*

Example 5.57. Let X be a discrete metric space (that is, one where the metric induces the discrete topology on X), we can take for example

$$d(x, y) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \ . \end{cases}$$

Then X is bounded in any case, but it is only compact if and only if it is finite. To see this, observe on the one hand that once X is finite, any open cover of X consists of finitely many sets (the power set of X being finite), hence compactness is satisfied no problem.

On the other hand, consider the open cover of X consisting of all one-element sets. As every point of X is contained in exactly one subset, this cover has no proper subcover. Therefore it possesses a finite subcover precisely if it is itself finite, i.e. X has finitely many elements.

Definition 5.58. Let $A \subseteq X$ be an arbitrary subset of the metric space (X, d); then we define the *distance of a point x from A* by

$$d(x, A) \stackrel{\text{def}}{=} \inf \{d(x, a) \mid a \in A\} \ .$$

In a similar vein, one defines that *diameter of A* to be

$$\operatorname{diam}(A) \stackrel{\text{def}}{=} \sup \{d(a_1, a_2) \mid a_1, a_2 \in A\} \ .$$

Furthermore, we set the *ϵ-neighbourhood of A to be*

$$\mathbb{B}(A, \epsilon) \stackrel{\text{def}}{=} \{x \in X \mid d(x, A) < \epsilon\} \ .$$

Proposition 5.59. *With notation as above, the function $d(\cdot, A) : X \to \mathbb{R}$ is continuous.*

Proof. Let $x, y \in X$ and $a \in A$ be arbitrary points. Observe that by definition, and the triangle inequality

$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a) ,$$

hence

$$d(x, A) - d(x, y) \leq d(y, a) .$$

By varying a all over A, we obtain that

$$d(x, A) - d(x, y) \leq \inf_{a \in A} d(y, a) = d(y, A) ,$$

that is,

$$d(x, A) - d(y, A) \leq d(x, y) .$$

After interchanging the roles of x and y, we arrive at

$$d(y, A) - d(x, A) \leq d(x, y) ,$$

which implies

$$|d(x, A) - d(y, A)| \leq d(x, y) .$$

It follows immediately that $d(\cdot, A)$ is continuous. $\square$

Exercise 5.60. *Prove the following properties of the distance function from a subset $A \neq \emptyset$.*

(1) *$d(x, A) \neq 0$ precisely if $x \in \overline{A}$.*
(2) *If the subset $A \subseteq X$ is compact, then there exists $a \in A$ for which*
$$d(x, A) = d(x, a) .$$
(3) *$\mathbb{B}(A, \epsilon) = \bigcup_{a \in A} \mathbb{B}(a, \epsilon)$.*
(4) *Let $A \subseteq X$ be a compact subset, $U \supseteq A$ an arbitrary open subset of X. Then there exists $\epsilon > 0$ such that $\mathbb{B}(A, \epsilon) \subseteq U$. Show that this does not hold in general if A is only assumed to be closed.*

Theorem 5.61. *Let (X, d) be a metric space. Then the following are equivalent.*

(1) *X is compact;*
(2) *X is limit point compact;*
(3) *X is sequentially compact;*
(4) *X is complete and totally bounded.*

The most difficult part is to show that sequential compactness/limit point compactness imply compactness. This depends largely on the following non-trivial result.

Lemma 5.62 (Lebesuge number lemma)**.** *Let $\mathcal{U}$ be an open covering of the compact metric space (X, d). Then there exists a positive constant $\delta > 0$ (depending on (X, d) and $\mathcal{U}$), such that for every subset $Z \subseteq X$ having diameter less than δ, there exists an element $U \in \mathcal{U}$ such that $Z \subseteq U$.*

Proof. Consider first the case when $X \in \mathfrak{U}$. Then any positive number is a Lebesgue number. Therefore we can assume without loss of generality that $X \notin \mathfrak{U}$.

Let $U_1, \ldots, U_n$ be a finite subcover of $\mathfrak{U}$, such a cover exists by the compactness of X. Set

$$C_i \stackrel{\text{def}}{=} X - U_i$$

for all $1 \leq i \leq n$, and define

$$f(x) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{i=1}^{n} d(x, C_i) \ .$$

The continuity of the distance function then implies that $f : X \to \mathbb{R}$ is a continuous function. We will now prove that $f(x) > 0$ for every point $x \in X$. Indeed, fix $x \in X$ arbitrary, and let $1 \leq i \leq n$ such that $x \in U_i$. Choose a positive real number ϵ such that $\mathbb{B}(x, \epsilon) \subseteq U_i$. Then $d(x, C_i) \geq \epsilon$ for every $1 \leq i \leq n$, and hence $f(x) \geq \dfrac{\epsilon}{n}$.

The function f being continuous, it takes on a minimal value $\delta = f(y)$, which is then bound to be strictly positive, since f is strictly positive at every point of X. We verify that this δ satisfies the requirements of being a Lebesgue number for X and $\mathfrak{U}$. To this end, consider an arbitrary subset $S \subseteq X$ of diameter $< \delta$, pick a point $x_0 \in S$. Observe that

$$B \subseteq \mathbb{B}(x_0, \delta) \ ,$$

and

$$\delta \ \leq \ f(x_0) \ \leq \ d(x_0, C_i)$$

for every $1 \leq i \leq m$. But then

$$\mathbb{B}(x_0, \delta) \subseteq U_{i_0} \ \in \ \mathfrak{U} \ ,$$

where i_0 is the index between 1 and n with the largest $d(x_0, C_i)$. $\square$

Lemma 5.63. *If X is sequentially compact, then the Lebesgue number lemma holds in X.*

Proof. Let $\mathfrak{U}$ be an open cover of X, suppose that $\mathfrak{U}$ does not have a Lebesgue number, that is, there does not exist $\delta > 0$ such that each subset of X with diameter less than δ would be contained in an element of $\mathfrak{U}$.

In this case, there exists a set C_n of diameter less than $\frac{1}{n}$ not contained in any element of $\mathfrak{U}$ for every natural number n. For every $n \in \mathbb{N}$, fix a point $x_n \in C_n$, and consider the sequence (x_n). By hypothesis, every sequence in X has a convergent subsequence, hence so does (x_n), let $(x_{n_k}) \to x$ denote such a subsequence.

Since $\mathfrak{U}$ covers X, there exists $U \in \mathfrak{U}$ containing x; as $U \subseteq X$ is open, there exists an open ball $\mathbb{B}(x, \epsilon) \subseteq U$. If k is large enough so that

$$\frac{1}{n_k} \ < \ \frac{\epsilon}{2} \ , \quad \text{and} \quad d(x_{n_k}, x) \ < \ \frac{\epsilon}{2} \ ,$$

then $C_{n_k} \subseteq \mathbb{B}(x_{n_k}, \frac{\epsilon}{2})$, and $\mathbb{B}(x_{n_k}, \frac{\epsilon}{2}) \subseteq \mathbb{B}(x, \epsilon)$, and so $C_{n_k} \subseteq U$, a contradiction. $\square$

Proof of Theorem 5.61. We verify to begin with that the first three characterizations are equivalent. It has been shown earlier that compactness implies limit point compactness.

Assume now that X is limit point compact, we aim at proving that it is sequentially compact as well. Let (x_n) be a sequence in X, consider the associated subset

$$A \stackrel{\text{def}}{=} \{x_n \,|\, n \in \mathbb{N}\} \ .$$

If $\#A < \emptyset$ then there exists a subsequence (x_{n_k}) which is constant, hence convergent. If A is infinite, then A has a limit point a. We will inductively construct a convergent subsequence of (x_n).

Let n_1 be a positive integer such that

$$x_{n_1} \in \mathbb{B}(x, 1) \ .$$

Assume that $x_{n_1}, \ldots, x_{n_{k-1}}$ have already been defined. Since the ball $\mathbb{B}(x, \frac{1}{k})$ has infinitely many points in common with A, there is an index $n_k > n_{k-1}$ with that property that

$$x_{n_k} \in B(x, \frac{1}{i}) \ .$$

Then the subsequence (x_{n_k}) converges to x as $k \to \infty$.

We will now prove that if X is sequentially compact, then it is complete and totally bounded. Let (x_n) be a Cauchy sequence in X. By sequential compactness, (x_n) has a subsequence (x_{n_k}) converging to some point $x \in X$. This means that for every $\epsilon > 0$, there exists a natural number N_ϵ such that

$$d(x_{n_k}), x) \ < \ \epsilon$$

whenever $n_k > N_\epsilon$. The definition of a Cauchy sequence provides us with a similar set of inequalities: for every $\epsilon > 0$ we have a natural number $M_\epsilon > 0$ such that

$$d(x_n, x_m) \ < \ \epsilon$$

provided $n, m > M_\epsilon$. Therefore,

$$d(x_n, x) \ \leq \ d(x_n, x_{n_k}) + d(x_{n_k}, x) \ < \ \frac{\epsilon}{2} + \frac{\epsilon}{2} \stackrel{\text{def}}{=} \epsilon$$

if $n > \max N_{\epsilon/2}, M_{\epsilon/2}$, hence the whole sequence (x_n) converges to x.

Next assume that X is sequentially compact. We prove that X is totally bounded. Suppose that it is not, and let $\epsilon > 0$ be such that X cannot be covered by a finite number of ϵ-balls. Then one can construct a sequence (x_n) in X, such that for every $m < n$, one has $d(x_m, x_n) \geq \epsilon$; that is, any two points in this sequence are at least ϵ distance apart. But then (x_n) cannot have a convergent subsequence.

With all this in mind, let $\mathcal{U}$ be an arbitrary open cover of X. Because the Lebesgue number lemma holds in X by Lemma 5.63, the open cover $\mathcal{U}$ has a Lebesgue number δ. By total boundedness we can cover X by a finite number of open balls of radius $\delta/3$. Since each of this balls has a diameter at most $\frac{2\delta}{3}$, every one of them lies in an element of $\mathcal{U}$. Picking such an element of the open cover $\mathcal{U}$ for each of these balls, we arrive at a finite subcover of $\mathcal{U}$.

Finally, we prove that the first three conditions are in fact equivalent to X being complete and totally bounded. Assuming (1)-(3) hold, X must be complete, since every Cauchy sequence has a convergent subsequence by sequential compactness. We have also seen above that sequential compactness implies total boundedness.

To finish off the proof, we show that completeness and total boundedness imply sequential compactness for X. Let (x_n) be an arbitrary sequence in X. The plan is first to construct a subsequence which is Cauchy, whence convergent. To this end, cover X by balls or radius 1. This is possible since X is totally bounded. Then at least one of these balls, call it B_1 contains infinitely many elements of the sequence. Let $I_1 \subseteq \mathbb{N}$ be the set of indices for which this holds.

Continuing this process in an inductive fashion, assuming the existence of an infinite set $I_{k-1} \subseteq \mathbb{N}$ for which $x_n \in B_{k-1}$ with the latter being an open ball with radius $\frac{1}{k-1}$, we can find an infinite set of positive integers $I_k \subseteq I_{k-1}$ for which there exists a ball B_k of radius $\frac{1}{k}$ with the property that $x_n \in B_k$ whenever $n \in I_k$.

Pick $n_1 \in I_1$, and given n_{k-1}, choose $n_k \in I_k$ such that $n_k > n_{k-1}$. Since all the sets I_k are infinite, we can always do this. By construction (more precisely, by virtue of the fact that $I_{k-1} \supseteq I_k$ for every k), for all $i, j \geq k$, the elements x_{n_i} and x_{n_k} belong to B_k, a ball of radius $\frac{1}{k}$. Therefore, the subsequence (x_{n_k}) is Cauchy, and since X is complete, it is convergent as well. $\square$

Corollary 5.64. *If (X,d) is a compact metric space, then for any $\epsilon > 0$ there can only be finitely many points $x_1, \ldots, x_n$ such that their pairwise distances are all at least ϵ.*

Definition 5.65. A function $f : (X, d_X) \to (Y, d_Y)$ between metric spaces is called an *isometry*, if for any $x, x' \in X$ one has

$$d_X(x, x') = d_Y(f(x), f(x')) .$$

Remark 5.66. We note here that in the literature an isometry if often required to be surjective.

Proposition 5.67. *Let (X,d) be a metric space, $f : X \to X$ an isometry. Then f is a continuous function. If in addition X is compact, then f is a homeomorphism.*

Proof. $\square$

Theorem 5.68. *Every metric space is locally compact.*

Exercise 5.69. *Let (X,d) be a metric space, $f : X \to X$ a contraction; that is, f is a continuous map for which there exists a real number $0 < c < 1$ such that for every $x, y \in X$ one has*

$$d(f(x), f(y)) < c \cdot d(x, y) .$$

Show that if X is compact, then f has a unique fixed point (i.e. a point $x \in X$ for which $f(x) = x$).

Exercise 5.70. *Let X be a compact topological space, $\{A_\alpha \mid \alpha \in I\}$ a collection of closed subsets of X, which is closed under finite intersections. If an open set $U \subseteq X$ contains $\cap_{\alpha\in I} A_\alpha$, then there exists an index $\alpha \in I$ such that $A_\alpha \subseteq U$.*

6. Quotient spaces

6.1. **Quotient topology.** Forming quotient spaces with respect to various structures (equivalence relations, group actions, etc.) is a fundamental tool in topology. It is in many ways the most cumbersome of the methods for constructing topologies we have seen.

Definition 6.1. Let (X, τ) be a topological space, Y a set $f : X \to Y$ a surjective function. We define a topology on Y called the *topology induced by* f or the *quotient topology* by specifying $V \subseteq Y$ to be open if and only if $f^{-1}(V) \subseteq X$ is open.

Note that Y is a bare set with no additional structure, f is a function of sets. The topology on Y induced by f is the largest (that is, containing the largest number of open sets) that makes f continuous.

Definition 6.2. Let X be a topological space, $\sim$ an equivalence relation on X, $Y \overset{\text{def}}{=} X/\sim$ the set of equivalence classes of $\sim$, $\pi : X \to Y = X/\sim$ the function sending every $x \in X$ to its equivalence class.

Then Y with the topology induced by π is called the *quotient space of X by $\sim$*.

It is important to remember that the quotient topology is a very tricky device, one has to be very careful when working with it. The point is, that while forcing various points in the space to 'get near' to each other, many other points might get near that we had not originally expected.

Here is an example.

Example 6.3. Consider the real line $\mathbb{R}$ with the equivalence relation $x \sim y$ if $x - y \in \mathbb{Q}$. Then $\mathbb{R}/\sim$ has uncountably many points, but has the trivial topology.

The next statement can be paraphrased as 'the quotient space of a quotient space is a quotient space'.

Proposition 6.4. *Let (X, τ) be a topological space, Y,Z sets,*

$$X \overset{f}{\twoheadrightarrow} Y \overset{g}{\twoheadrightarrow} Z$$

surjective functions. Let σ be the topology on Y induced by f, τ be the topology on Z induced by g from (Y, σ), and let μ' be the topology on Z induced by $g \circ f$ from (X, τ). Then $\mu = \mu'$, that is, the two induced topologies on Z coincide.

Proof. □

Definition 6.5. A continuous map of topological spaces $f : X \to Y$ is called an *indentification map* if it surjective, and Y has the quotient topology with respect to f as a function.

Identification maps have the following interesting universal property.

Proposition 6.6. *A surjective continuous map $f : X \to Y$ between topological spaces is an identification map if and only if for every function $g : Y \to Z$ the composition $g \circ f$ is continuous precisely if g is continuous.*

Proof. First let $f : X \to Y$ be an identification map, $g : Y \to Z$ an arbitrary function. If g is a continuous map, then so is the composition $g \circ f$. Assume now that $g \circ f$ is continuous, i.e. for every $V \subseteq Z$ open, $f^{-1}(g^{-1}(V) = (g \circ f)^{-1}(V) \subseteq X$ is open. Consider $g^{-1}(V) \subseteq Y$. Since f is an identificication map, Y has the quotient topology with respect to f. This means that $g^{-1}(V) \subseteq Y$ is open if and only if $(g \circ f)^{-1}(V) \subseteq X$ is open, hence we are done.

Assume now that f has the property in the proposition, let (Y, σ) denote the topology of Y. Let us specialize to the case $Z = Y$ *as sets.* Now take the function

$$g = \mathrm{id} : Y \longrightarrow Z ,$$

and give Z the topology induced by g from Y. Observe that the function $g \circ f : X \to Z$ is continuous, as

$$(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V) = f^{-1}(\mathrm{id}^{-1}(V) = f^{-1}(V)$$

is open in X for every open set $V \subseteq Z$ in the induced topology from id. Therefore g is continuous as well by the universal property. But the function g^{-1} is continuous as well, since $(g \circ f) \circ g^{-1} = f$, hence g is a homeomorphism, $Y \approx Z$, and f an identification map. Then $j \circ f = g \circ i$ also as required, hence the diagram above commutes. □

The following extended example is very important.

Example 6.7 (Real projective plane $\mathbb{RP}^2$)**.** The real projective plane $\mathbb{RP}^2$ is traditionally defined as the quotient $\mathbb{R}^3 - 0/\sim$, where $x \sim y$ if and only if $x = \lambda y$ for some nonzero $\lambda \in \mathbb{R}$. Here we will take an alternate route leaving the proof that the former definition gives a homeomorphic result.

Let $\mathbb{S}^2 \subseteq \mathbb{R}^3$ denote the unit sphere in Euclidean three-space; we define the equivalent relation $\sim$ on $\mathbb{S}^2$ by

$$x \sim y \iff x = -y ,$$

that is, we make antipodal points equivalent. The equivalence classes are the two-element sets $\{x, -x\}$.

Consider $i : \mathbb{D}^2 \hookrightarrow \mathbb{S}^2$ embedded as the northern hemisphere. Define the equivalence relation $\asymp$ on $\mathbb{D}^2$ in the following fashion: every point in the interior forms its own equivalence class, points on the boundary (one the 'equator' so to speak) form two-element classes along with their antipodal pairs.

Look at the diagram

$$
\begin{tikzcd}
\mathbb{D}^2 \arrow[r, "i"] \arrow[d, "f"'] & \mathbb{S}^2 \arrow[d, "g"] \\
\mathbb{D}^2/\asymp \arrow[r, "j"] & \mathbb{S}^2/\sim
\end{tikzcd} \quad . \tag{2}
$$

Here f and g are the appropriate identification maps, and j the induced function

$$j : \mathbb{D}^2/\asymp \longrightarrow \mathbb{S}^2/\sim\ ,$$

the unique function making the diagram commute, whose existence we need yet to prove.

Set $j([x]) \stackrel{\text{def}}{=} i(x)$ for every $x \in \mathbb{D}^2/\asymp$. If x is in the strict upper hemisphere (i.e. the third coordinate of x is positive), then the equivalence class $[x]$ consists of one element only, hence $j([x])$ is automatically well-defined as the image of x in $\mathbb{S}^2/\sim$. If x is on the equator, then $[x] = \{x, -x\}$, however, both points are mapped to elements of the same equivalence class of $\sim$, hence again, $j([x])$ is well-defined, and we have a function $j : \mathbb{D}^2/\asymp\to \mathbb{S}^2/\sim$. It can also be seen quickly that j is one-to-one and onto.

Next, the continuity of j. Let $U \subseteq \mathbb{S}^2/\sim$ be an open subset, then $g^{-1}(U) \subseteq \mathbb{S}^2$ is open by definition of the quotient topology. The inclusion i is continuous, hence

$$(gi)^{-1}(U) = i^{-1}(g^{-1}(U)) \subseteq \mathbb{D}^2$$

is open. By the commutativity of the diagram 2,

$$f^{-1}(k^{-1}(U)) = (kf)^{-1}(U) = (gi)^{-1}(U) \subseteq \mathbb{S}^2$$

is again open. This means that $j^{-1}(U) \subseteq \mathbb{D}^2/\asymp$ is open by the definition of the quotient topology, hence j is a continuous map.

Observe that $\mathbb{D}^2/\asymp$ is compact (as $\mathbb{D}^2$ is, being a closed and bounded subset of $\mathbb{R}^3$), $\mathbb{S}^2$ can easily be seen to be Hausdorff, therefore the continuous bijection $j : \mathbb{D}^2/\asymp\to S^2/\sim$ is a homeomorphism.

Example 6.8 (Torus)**.** Let $X = [0, 1] \times [0, 1] \subseteq \mathbb{R}^2$ with the subspace topology. Consider the following partition of X giving an equivalence relation $\sim$:

- one-point sets $\{(x, y)\}$ for every point (x, y) with $0 < x, y < 1$,
- $\{(x, 0), (x, 1)\}$ if $0 < x < 1$,
- $\{(0, y), (1, y)\}$ if $0 < y < 1$,
- $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

The corresponding quotient space $X/\sim$ is called the *torus*.

An often used special case of the quotient topology is the 'collapsing' of a subspace.

Definition 6.9. Let X be a topological space, $A \subseteq X$ arbitrary. Then X/A denotes the quotient space obtained via the equivalence relation whose classes are A, and the sets $\{x\}$, $x \in X - A$.

We collapse A to a point, so to speak.

Example 6.10. Consider the cylinder $\mathbb{S}^n \times I$ (with $I = [0,1]$ being the unit interval), and define

$$\begin{array}{rcl} f : \mathbb{S}^n \times I & \longrightarrow & \mathbb{D}^{n+1} \\ (x,t) & \mapsto & tx \ . \end{array}$$

Then f carries $\mathbb{S}^n \times \{0\}$ to the origin, hence f 'factors through' $\mathbb{S}^n \times I/\mathbb{S}^n \times \{0\}$. This means that there exists a continuous map g

$$\begin{tikzcd} \mathbb{S}^n \times I \arrow[r, "f"] \arrow[dr, "\pi"'] & \mathbb{D}^{n+1} \\ & \mathbb{S}^n \times I/\mathbb{S}^n \times \{0\} \arrow[u, dashed, "g"'] \end{tikzcd} \ .$$

One can show that g is bijective, which, together with the facts that its source is compact and its target Hausdorff, gives that g is a homemorphism.

Exercise 6.11. *Show that* $\mathbb{D}^n/S^{n-1} \approx \mathbb{S}^n$.

Definition 6.12. If $A \subseteq X$ is an arbitrary subspace, $\sim$ an equivalence relation on X, then the *saturation on A with respect to* $\sim$ is the subspace

$$\{x \in X \mid \exists a \in A \ x \sim a\} \ .$$

Proposition 6.13. *Let $A \subseteq X$ be an arbitrary subspace, $\sim$ an equivalence relation on X such that every equivalence class intersects A. Then the induced map*

$$k : A/\sim \longrightarrow X/\sim$$

is a homeomorphism if the saturation of every open set of A is open in X.

Proof. Consider the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"] & X \arrow[d, "g"] \\ A/\sim \arrow[r, "k"] & X/\sim \end{tikzcd} \ ,$$

where f and g are the respective identification maps. If $U \subseteq A/\sim$ is open, then $g^{-1}(k(U)$ is the saturation of $f^{-1}(U)$. Moreover, by definition U ??? $k(U) \subseteq X/\sim$ open. Also, k is bijective and continuous. □

Definition 6.14. Let X,Y be topological spaces, $A \subseteq X$ a closed subset, $f : A \to Y$ a continuous map. Then *attaching* Y *to* X *along* f is defined as

$$X \coprod Y/\sim \ ,$$

where $\sim$ is the equivalence relation generated by the pairs $a \sim f(a)$ for all $a \in A$.

Remark 6.15. Note that the equivalence relation generated by all pairs $a \sim f(a)$, $a \in A$ consists of the following pairs: for $u, v \in X$,

$$u \sim v \iff \begin{cases} u = v\ , & \text{or} \\ u, v \in A \text{ and } f(u) = f(v)\ , & \text{or} \\ u \in A \text{ and } v = f(u) \in Y\ . \end{cases}$$

Example 6.16 (The mapping cone and the mapping cylinder)**.**

Lemma 6.17. *Let $f : X \to Y$ be a continuous closed surjective map. Then f is a quotient map.*

Proof. If $U \subseteq Y$ is open, then so is $f^{-1}(U) \subseteq X$ by the continuity of f. In the other direction, let $V \subseteq Y$ an arbitrary subset such that $f^{-1}(V) \subseteq X$ is open. Then $f(X - f^{-1}(V)) \subseteq Y$ is closed, and since f is surjective,

$$f(X - f^{-1}(V)) = V\ .$$

□

Here we will present a condition under which the quotient space with respect to an equivalence relation is Hausdorff. This fact will be used when we will deal with topological group actions.

Proposition 6.18. *Let X be a topological space, $\sim$ an equivalence relation on X. If $X/\sim$ is Hausdorff, then the graph G of $\sim$ is closed in $X \times X$. If the relation $\sim$ is open as well, then this condition is also sufficient.*

Proof. As usual, we denote the canonical map taking an element of X to its equivalence class by $\pi : X \to X/\sim$. With this notation G is the inverse image of $\Delta \subseteq Xsim \times X/\sim$ under the continuous map $\pi \times \pi : X \times X \to Xsim \times X/\sim$. Hence, if $X/\sim$ is Hausdorff, then Δ is closed, but then so is $G \subseteq X \times X$.

If $\sim$ is open, then

$$X/\sim \times X/\sim \approx (X \times X)/(\sim \times \sim)$$

in which case Δ can be identified with the image of G (which is saturated with respect to $\sim \times \sim$. Therefore Δ_X is closed, so X is Hausdorff. □

6.2. **Group actions on topological spaces.** One particularly common form of quotient spaces occurs when a group acts on some topological space. In order to be able to describe this phenomenon in detail, first we need to recite group actions on sets. Although this is not at all necessary in general, here we will first restrict our attention to the case when the group carries no geometric information, ie. it has the discrete topology.

For starters, we discuss the set-theoretic case.

Definition 6.19. Let X be an arbitrary set, G a group. A *left action of G on X* is a set function

$$\alpha : G \times X \longrightarrow X$$

such that

$$\alpha(1, x) = x$$

for every $x \in X$, and

$$\alpha(g, \alpha(h, x)) = \alpha(gh, x)$$

for every $g, h \in G$. When it is clear from the context, the action α is often suppressed, one simply writes $g.x$ or just gx for $\alpha(g, x)$.

Note the following simple but important consequence of the definition: since $1 \in G$ acts as the identity function on X, the elements g and g^{-1} act as mutually inverse self-maps:

$$\begin{aligned} g^{-1}.(g.x) &= (g^{-1}g).x = 1.x = x \\ g.(g^{-1}.x) &= (gg^{-1}).x = 1.x = x \ . \end{aligned}$$

Therefore, for every $g \in G$, the function $\alpha(g, \cdot) : X \to X$ is bijective.

Whence, one can restate the notion of a left-action of a group in the following way. Let $\mathrm{Aut}(X)$ denote the group of bijective self-maps of the set X, with composition of functions as multiplication, the identity function as identity element, and the inverse function serving as the inverse element in the group. Then, a left action of G on X is a group homomorphism

$$\rho : G \longrightarrow \mathrm{Aut}(X) \ .$$

That the two definitions coincide follows from the observation that $(gh).x = g.(h.x)$ amounts to the same as $\rho(gh) = \rho(g) \circ \rho(h)$.

Exercise 6.20. *Check the details of the previous paragraphs very carefully.*

To elucidate the notion of group actions on sets, we will consider a handful of examples.

Example 6.21. First we will consider the simples examples. Let X be any set. Then the group of bijections $\mathrm{Aut}(X)$ acts on X by the *evaluation action*:

$$\begin{aligned} \mathrm{ev} : \mathrm{Aut}(X) \times X &\longrightarrow X \\ (\phi, x) &\mapsto \phi(x) \ . \end{aligned}$$

In an equally simplistic way, if X is a set, G an arbitrary group, then the *trivial action* is given by

$$\begin{aligned} \mathrm{triv} : G \times X &\longrightarrow X \\ (g, x) &\mapsto x \ . \end{aligned}$$

Example 6.22. Consider $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$. The antipodal action taking $x \mapsto -x$ can be described in terms of an action of the two-element group Z_2 on $\mathbb{S}^2$: the only element different from 1 will take x to $-x$. The interested reader should check the properties required for this assignment to be a group action are indeed satisfied.

Example 6.23. Let now $X = \mathbb{R}^n$, and fix a basis $e_1, \dots, e_n$ of $\mathbb{R}^n$. Consider

$$L \stackrel{\text{def}}{=} \left\{ \sum_{i=1}^{n} a_i e_i \,|\, a_i \in \mathbb{Z}\,,\, \forall 1 \leq i \leq n \right\} \ ,$$

which is called the *lattice generated by* $e_1, \dots, e_n$. Let each $\lambda \in L$ act on X by the translation

$$\alpha_\lambda(x) \ = \ \lambda + x \ .$$

Again, it is easily verified that we have defined a group action on X:

$$\begin{aligned} 0 + x \ &= \ x \\ \lambda + (\lambda' + x) \ &= \ (\lambda + \lambda') + x \ . \end{aligned}$$

Example 6.24. Consider the translation action of the additive group of $\mathbb{Z}$ on $\mathbb{R}$. Then we can identify the quotient (as a set, later we will see that this is true in the sense of topology as well) $\mathbb{R}/\mathbb{Z}$ with the unit circle $\mathbb{S}^1 \subseteq \mathbb{R}$ using trigonometry. For any $t \in \mathbb{R}$ we map

$$t \ \mapsto \ (\cos(2\pi t), \sin(2\pi t)) \ .$$

By periodicity of the trigonometric functions, this association depends exactly on the $\mathbb{Z}$-orbit of the point t (in other words: the respective images of $t, t' \in \mathbb{R}$ are equal if and only they lie in the same $\mathbb{Z}$-orbit, that is to say, iff $2\pi(t - t') \in \mathbb{Z}$). This mapping gives a well-defined bijection from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{S}^1$.

Note that it is also customary to use the action of the additive group $2\pi\mathbb{Z}$ on $\mathbb{R}$, which gives a bijection from $\mathbb{R}/2\pi\mathbb{Z}$ to $\mathbb{S}^1$.

Definition 6.25. Let $\alpha : G \times X \to X$ be a left action on the set X. The *G-orbit* of $x \in X$ is defined as

$$G^x \stackrel{\text{def}}{=} \{\alpha(g, x) \,|\, g \in G\} \ .$$

For a subset of points $S \subseteq X$, the *G-translate of S* is

$$G.S \stackrel{\text{def}}{=} \{g.s \,|\, g \in G, s \in S\} \ .$$

The *stabilizer or isotropy subgroup* of a point $x \in X$ is

$$G_x \stackrel{\text{def}}{=} \{g \in G \,|\, g.x = x\} \ .$$

The *set-theoretic quotient* X/G is the set of G-orbits of X, with the quotient map

$$\begin{aligned} \pi : X \ &\longrightarrow \ X/G \\ x \ &\mapsto \ G\text{-orbit of } x \ . \end{aligned}$$

If X, X' are two sets both endowed with left G-actions, then a function $f : X \to X'$ is called *G-equivariant*, provided

$$f(g.x) \;=\; g.f(x)$$

for every point $x \in X$ and group element $g \in G$.

A subset $S \subseteq X$ is called G-stable (or α-stable, if we want to be more precise), if $G.S = S$, that is, if $g.s \in S$ for every $g \in G$ and $s \in S$.

Remark 6.26. G-stable sets are exactly the ones to which one can restrict the given G-action. In the definition of a G-stable set, it is enough to require that $G.S \subseteq S$, since $S \subseteq G.S$ in any case.

Exercise 6.27. *Prove that the G-orbits of X form the equivalence classes of the equivalence relation $x \sim_G y$ if and only if there exists $g \in G$ such that $g.x = y$.*

Exercise 6.28. *Work out the equivalence classes of the given G actions in all the examples above.*

Remark 6.29. Any G-equivariant function $f : X \to X'$ carries the G-orbit of an element $x \in X$ into the G-orbit of $f(x) \in X'$, hence there exists a well-defined function

$$\overline{f} : X/G \longrightarrow X'/G$$

sending $G^x \in X/G$ to $G^{f(x)} \in X'/G$. Also, note that the the functions f and $\overline{f}$ are compatible with the projections π,π' giving rise to a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & X' \arrow[d, "\pi'"] \\ X/G \arrow[r, "f'"] & X'/G \end{tikzcd}$$

with $\overline{f} \circ \pi = \pi' \circ f$. We call $\overline{f}$ the *function induced by f*.

Example 6.30. Let $X = \mathbb{R}$, and $G = 2\pi\mathbb{Z}$ acting by additive translations. Consider the function

$$\begin{array}{rcl} f : \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x + c \end{array}$$

for some fixed $c \in \mathbb{R}$ (where both copies of $\mathbb{R}$ are taken with the *same* G-action. The function f is G-equivariant, and the induced map

$$\overline{f} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$$

is rotation thru an angle of c.

Let us now reintroduce topology into the picture. In what follows the group G will always have the discrete topology.

Definition 6.31. Let X be a topological space, G a discrete group acting on X via $\alpha : G \times X \to X$ ($G \times X$ is given the product topology). The left-action α is called *continuous*, if it is continuous as a function.

The action α is *free*, if $G_x = \{1\}$ for every $x \in X$; it is said to be *properly discontinuous* if every element $x \in X$ admits an open neighbourhood $U_x \subseteq X$, such that

$$g.U_x \cap U_x = \emptyset$$

for all but finitely many $g \in G$.

Naturally, all actions by finite groups are properly discontinuous.

Remark 6.32. The fact that $\alpha : G \times X \to X$ as above is continuous is equivalent to requiring that

$$\alpha(g, \cdot) : X \to X$$

is a continuous map for every $g \in G$.

Properly discontinuous actions are particularly useful in the context of locally Hausdorff spaces.

Exercise 6.33. *Let $\alpha : G \times X \to X$ be a free and properly discontinuous action, $S \subseteq X$ is a G-stable subset equipped with the subspace topology. Show that the restricted action $\alpha : G \times S \to S$ is also free and properly discontinuous.*

Definition 6.34. A topological space X is *locally Hausdorff*, if every point $x \in X$ has a neighbourhood which is Hausdorff in the subspace topology inherited from X.

Proposition 6.35. *Let X be a locally Hausdorff topological space, $\alpha : G \times X \to X$ a properly discontinuous left-action. Then every point $x \in X$ has an open neighbourhood U_x such that for every $g \in G$*

$$g.U_x \cap U_x \neq \emptyset \iff g.x = x \ .$$

Consequently, if the action α is free as well, then

$$g.U_x \cap U_x \neq \emptyset \iff g = 1 \ .$$

Proof. Let V_x be a neighbourhood of $x \in X$ such that

$$g.V_x \cap V_x \neq \emptyset$$

holds for only finitely many group elements $g_1, \dots g_m$. By shrinking V_x is necessary, we can assume without loss of generality that $V_x \subseteq X$ is open and Hausdorff.

We show that for every $1 \leq i \leq m$ such that $g_i.x \in V_x \setminus \{x\}$, there exists an open subset $U_i \subseteq V_x$ with $g_i.U_i \cap U_i = \emptyset$. By the Hausdorff property of V_x, whenever $g_i.x \in V_x \setminus x$, there exist disjoint open subsets $V_i, V_i' \subseteq V_x$ around x, and $g_i.x$, respectively. By continuity of the action of g_i on X, there is an open set $x \in W_i \subseteq X$ for which $g_i.W_i \subseteq V_i'$. Thus, $U_i \stackrel{\text{def}}{=} W_i \cap V_i$ is disjoint from V_i', and satisfies $g_i.U_i \subseteq V_i'$, hence $U_i \cap g_i.U_i = \emptyset$.

With this in hand, we can take

$$U_x \stackrel{\text{def}}{=} U_1 \cap \cdots \cap U_m \ .$$

□

Exercise 6.36. *Prove the following coverse: if $\alpha : G \times X \to X$ is a left-action such that for every $x \in X$ there exists a neighbourhood U_x around x for which*

$$g.U \cap U \neq \emptyset \iff g = 1 \ ,$$

then α is free and properly discontinuous.

Example 6.37. Fix a non-zero natural number m, set $X = \mathbb{R}^2$. Let $G = \mathbb{Z}/m\mathbb{Z}$ be the group of modulo m residue classes with respect to addition. We let the element $a \bmod m \in G$ act on $\mathbb{R}^2$ via counterclockwise rotation by an angle of $\frac{2\pi a}{m}$. As it is known, this action does not actually depend on the representative $a \in \mathbb{Z}$, only on the residue class $a \bmod m$.

For any nonzero $x \in \mathbb{R}^2$, the orbit of x consists of m distinct points on the circle $\mathbb{S}_x$ of radius $\|x\|$ centered at the origin. An open ball $\mathbb{B}(x, \epsilon)$ of sufficiently small radius the translates of $\mathbb{B}(x, \epsilon)$ by non-identity elements of G are disjoint from $\mathbb{B}(x, \epsilon)$. In other words, rotating $\mathbb{B}(x, \epsilon)$ about the origin by an angle of $\frac{2\pi a}{m}$ produces a set disjoint from $\mathbb{B}(x, \epsilon)$ except when $m|a$. So, on $\mathbb{R}^2 \setminus (0, 0)$, the rotation action of $\mathbb{Z}/m\mathbb{Z}$ is free and properly discontinuous.

However, the situation differs largely at $(0, 0) \in \mathbb{R}^2$ if $m > 1$. The origin is fixed by every element of G, hence every one of its neighbourhoods meets every its translates by any element of G. Therefore, the action of G on $\mathbb{R}^2$ is *not* free, although it is still properly discontinuous.

Example 6.38. Another simple example is the so-called split action. Let X' be a topological space, G any group, and consider the product $X \stackrel{\text{def}}{=} G \times X'$ (equipped with the product topology, where G is taken to be discrete, as always). One can check that

$$X = \coprod_{g \in G} X' \ ,$$

that is, X is by definition the disjoint union of copies of X' indexed by elements of the group G. We will say that $\{g\} \times X' \subseteq X$ is the g^{th} *copy of* X'.

The split action of G on X is defined by left multiplication:

$$g.(h, x) \stackrel{\text{def}}{=} (gh, x) \ ,$$

and is quickly seen to be free and properly discontinuous. One can identify the quotient X/G with X' via the second projection map $X = G \times X' \to X'$.

In general, for an arbitrary topological space X and a group G, a continuous left G-action is defined to be *split*, if there exists a topological space Y and a G-equivariant homeomorphism from X to $G \times Y$ carrying the G action given on X to the split action on $G \times Y$. In other words, this amounts to requiring X to contain an open subset $Y \subseteq X$ such that the open subsets $g.Y$ for $g \in G$ are pairwise disjoint and cover X.

Theorem 6.39. *Let X be a locally Hausdorff topological space, $\alpha : G \times X \to X$ a properly discontinuous and free action on X. Then the quotient map*

$$\pi : X \longrightarrow X/G$$

is a local homeomorphism. If $U \subseteq X$ is an open subset such that $g.U \cap U = \emptyset$ for all $g \neq 1 \in G$, then the action of G on $\pi^{-1}(\pi(U))$ is split.

Proof. First we address the issue of π being a local homeomorphism. Let $x \in X$ be arbitrary. By the condition that the action of G is free and properly discontinuous, there exists an open neighbourhood $U_x \subseteq X$ of x such that

$$g.U_x \cap U_x = \emptyset$$

for all $g \in G$ different from $1 \in G$. If $y, z \in U_x$ are to have the same image in X/G, then they belong to the same G-orbit, therefore $g.y = z$ for some $g \in G$. This implies that $g.U_x \cap U_x \neq \emptyset$, so one must have $g = 1 \in G$, and consequently, $y = z$ We can conclude, that U_x injects into X/G.

Since $\pi|_{U_x}$ is by construction surjective onto its image, all that is left to show is $\pi|_{U_x}$ is an open map, that is, for any open set $U \subseteq U_x$ the image of $U \subseteq X/G$ should be open. To this end, we need to verify that

$$\pi|_{U_x}^{-1}(\pi|_{U_x}(U) \subseteq X$$

is open. This follows from the fact that

$$\pi|_{U_x}^{-1}(\pi|_{U_x}(U)) = \bigcup_{g \in G} g.U \ ,$$

since all sets $g.U \subseteq X$ are open (the maps $x \mapsto g.x$ are homeomorphisms of X onto itself).

Now we can deal with the statement that the action of G is locally split. We will show that for any open set $U \subseteq X$ with the property that

$$g.U_x \cap U_x = \emptyset$$

for all $g \neq 1$, we have that the restricted action map

$$\begin{array}{rcl} \alpha : G \times U & \longrightarrow & \pi^{-1}(\pi(U)) \\ (g.x) & \mapsto & g.x \end{array}$$

is a homeomorphism.

Note first that α is surjective: if $x \in \pi^{-1}(\pi(U))$, then there exists $u \in U$ with $\pi(x) = \pi(u)$, that is, x and u lie in the same G-orbit, hence there exist $g \in G$ with $g.u = x$. This just says that $\alpha(g, u) = x$.

Next, assume that

$$g_1.u_1 = g_2.u_2$$

for some $g_1, g_2 \in G$ and $u_1, u_2 \in U$. Then

$$g_2^{-1}.(g_1.u_1) \;=\; u_2 \;,$$

which, by the associativity propery of group actions amounts to

$$(g_2^{-1}g_1).u_1 \;=\; u_2 \;.$$

Therefore $(g_2^{-1}g_1).U \cap U \neq \emptyset$, and so $g_2^{-1}g_1 = 1 \in G$ by the fact that α is free and properly discontinuous; hence $g_1 = g_2$ and $u_1 = u_2$. This implies that α is injective.

Since the action is free and properly discontinuous, the translates $g.U$ are disjoint and open in X, and they cover $\pi^{-1}(\pi(U))$. Hence, as a topological space, we have the disjoint union decomposition

$$\pi^{-1}(\pi(U)) \;=\; \coprod_{g\in G} g.U \;.$$

The bijective map α carries the open subset $\{g\}\times U \subseteq G\times U$ (homeomorphic to U) homeomorphically to the subset $g.U \subseteq \pi^{-1}(\pi(U))$. This means that α is a bijective map which restricts to homeomorphisms on respective collections of disjoint open sets covering the spaces $G\times U$ and $\pi^{-1}(\pi(U))$. But such a map is a homeomorphism. □

Lemma 6.40. *Let $\alpha : G \times X \to X$ be a free and properly discontinuous action on a locally Hausdorff topological space X. Then the quotient X/G is Hausdorff if and only if the image of the map*

$$\begin{array}{rcl} \alpha^+ : G \times X & \longrightarrow & X \times X \\ (g,x) & \mapsto & (g.x, x) \end{array}$$

is closed in $\times X$.

Proof. This is an elegant application of the diagonal criterion for the Hausdorff property. According to this, X/G is Hausdorff if and only if the image of

$$\Delta_{X/G} : X/G \longrightarrow X/G \times X/G$$

is closed. By the properties of the quotient topology, the quotient map $\pi : X \to X/G$ is open, hence so is the continuous surjective map

$$\pi \times \pi : X \times X \longrightarrow X/G \times X/G \;.$$

Hence, X/G has closed diagonal image precisely if the preimage of the diagonal in $X \times X$ is closed as a subset of $X \times X$. Observe that a point $(x,y) \in X \times X$ gets mapped to the diagonal of X/G if and only if x and y have the same image in X/G, in other words, whenever there exists $g \in G$ with $g.x = y$. But then (x,y) is in the image of the action map $(g,x) \mapsto (g.x, x)$. □

Proposition 6.41. *Let $S \subseteq X$ be a G-stable subset. Then the induced map of quotient sets*

$$\iota : S/G \longrightarrow X/G$$

is a homeomorphism onto its image. The image $\iota(S/G) \subseteq X/G$ is open/closed/locally closed precisely if $S \subseteq X$ is open/closed/locally closed.

The function $S \mapsto S/G$ is a bijection between the collection of G-stable subsets of X, and the power set of X/G.

Proof. Note, as a start, that ι is necessarily injective: if $s_1, s_2 \in S$ belong to the same G-orbit in X, then they are in the same G-orbit in S as well.

Next, pick a point $x \in X$. If x is in the G-orbit of a point $s \in S$, then $x \in S$, as S was chosen to be G-stable (hence a disjoint union of full orbits). Therefore, if we denote the quotient map by $\pi : X \to X/G$, then

$$S = \pi^{-1}(\pi(S)) = \pi^{-1}(S/G) .$$

Thus, we can see that the mapping $S \mapsto S/G$ is injective. Conversely, if $T \subseteq S/G$ is an arbitrary subset, then

$$S \stackrel{\text{def}}{=} \pi^{-1}(T)$$

is a G-stable subset of X, because $\pi(g.x) = \pi(x)$ for every $x \in X, g \in G$; in addition $S/G = T$ as subsets of X/G. Thus, we have proved the last statement of the proposition.

Let now $S \subseteq X$ be a G-stable subset, we will show that S is open/closed if and only if S/G is open/closed. Since the quotient map π is continuous and $S = \pi^{-1}(S/G)$, the subspace $S \subseteq X$ is open/closed provided S/G was. Note that the same conclusion holds in the case when S/G is assumed to be locally closed. If $S \subseteq X$ is open, then so is $\pi(S) = S/G$, as π is an open map. For the case when S is closed, note that the complement $X \setminus S$ of S is also G-stable, moreover it is open, hence

$$S/G = X/G \setminus \pi(X \setminus S)$$

is closed in X/G (the equality in the above formula comes from the surjectivity of π. We can conclude that ι is an open and closed map. To see that it is a homeomorphism, there is a bit more work to do.

First, ι is continuous, as composition with the quotient map $\pi|_S : S \to S/G$, which is a local homeomorphism yields a continuous map $S \to X/G$:

$$\begin{tikzcd} S \arrow[r] \arrow[d, "\pi|_S"'] & X \arrow[d, "\pi"] \\ S/G \arrow[r, "\iota"] & X/G . \end{tikzcd}$$

For ι to be a homeomorphism onto its image one needs that every open set $V \subseteq S/G$ is the inverse image of an open set in X/G. Observe that V is the image of a G-stable open set $U \subseteq S$, which just means that there exists an open set $W \subseteq X$ with $U = W \cap S$. That is, V is simply $S/G \cap \pi(W)$.

To finish the proof, let $S \subseteq X$ be a G-stable locally closed subset. In the process of proving that S/G is locally closed as well, we face the following difficulty: upon writing $S = U \cap F$ as an intersection of a open set $U \subseteq X$ and a closed set $F \subseteq X$, we might find that none of U and F is G-stable. The way around this problem is as follows: in any case we have $\overline{S} \subseteq F$, hence $S = U \cap \overline{S}$. Note that $\overline{S} \subseteq X$ is G-stable. The bijective correspondence between G-stable subsets in X and subsets of X/G, and likewise for closed sets in each implies that the closed subset

$$\overline{S}/G \;=\; \pi(\overline{S}) \;\subseteq\; X/G$$

is exactly the minimal closed subset containing S/G, that is, $\overline{S/G}^{X/G}$.

Since $S \subseteq \overline{S}$ is open, applying the proposition (the part of it which we have proven) to the space $\overline{S}$ equipped with its inherited free and properly discontinuous G-action we obtain that $S/G \subseteq \overline{S}/G$ is an open subset. Note that $\overline{S}/G$ maps homeomorphically onto its image, thus $S/G \subseteq X/G$ is an open subset of a closed subset of X/G, that is, $S/G \subseteq X/G$ is locally closed. $\square$

7. Homotopy

From now on we will slowly venture into the realm of algebraic topology. The following notion is of central importance. In this section I always denotes the unit interval $[0,1]$.

Definition 7.1. Let X,Y be topological spaces. A *homotopy of maps from X to Y* is a continuous map

$$F : X \times I \longrightarrow Y\ .$$

Two maps $f_0, f_1 : X \to Y$ are called *homotopic*, if there exists a homotopy of maps $F : X \times I \to Y$ such that for every $x \in X$

$$F(x,0) \;=\; f_0(x) \quad \text{and} \quad F(x,1) \;=\; f_1(x)\ .$$

The notation is $f \simeq g$, where we will typically suppress the actual homotopy F.

If two maps are homotopic, then a homotopy between them is far from unique. In fact, as we will see later, if there is one homotopy between two maps, then there will be lots.

Lemma 7.2. *Being homotopic is an equivalence relation. More precisely, let $f, g, h : X \to Y$ be maps, then*

(1) $f \simeq f$,
(2) if $f \simeq g$, *then* $g \simeq f$,
(3) if $f \simeq g$ *and* $g \simeq h$, *then* $f \simeq h$.

Proof. In all three cases we will exhibit a concrete homotopy showing that the required relation holds. For reflexivity, take the constant homotopy, ie. $F(x,t) \stackrel{\text{def}}{=} f(x)$ for all x and t. Then obviously $f_0 = f_1 = f$.

For symmetry, let $F : X \times I \to Y$ be a homotopy, we will turn F around by specifying

$$\widetilde{F}(x,t) \stackrel{\text{def}}{=} F(x, 1-t)$$

for all $x \in X$ and $t \in I$. Then $\widetilde{F}$ is again a continuous function from $X \times I \to Y$ (being the composition of the continuous functions $t \mapsto 1-t$ and F), hence a homotopy from $F(x,0)$ to $F(x,1)$. By construction however $F(x,0) = g$, and $F(x,1) = f$, hence $g \simeq f$.

Let us now treat transitivity. Again, $f \simeq g$ and $g \simeq h$ mean that there exists homotopies $F : X \times I \to Y$ and $G : X \times I \to Y$ from f to g, and from g to h, respectively. We want to combine somehow F and G to provide a homotopy from f to g. Here is one way to do it:

$$(F \star G)(x,t) = \begin{cases} F(x, 2t) & \text{if } 0 \le t \le \frac{1}{2} \\ G(x, 2t-1) & \text{if } \frac{1}{2} < t \le 1 \ . \end{cases}$$

Note that since $F(x,1) = g(x) = G(x,0)$, the function $F \star G : X \times I \to Y$ is continuous, hence indeed provides a homotopy from f to h. $\square$

The constructions used in the proof merit to be defined separately.

Definition 7.3. Let $F, G : X \times I \to Y$ be homotopies with $F(x,1) = G(x,0)$ for all $x \in X$. Then we can define the *inverse of* F to be

$$\widetilde{F}(x,t) \stackrel{\text{def}}{=} F(x, 1-t) \ ,$$

and the *concatenation of* F *and* G as

$$(F \star G)(x,t) = \begin{cases} F(x, 2t) & \text{if } 0 \le t \le \frac{1}{2} \\ G(x, 2t-1) & \text{if } \frac{1}{2} < t \le 1 \ . \end{cases}$$

The inverse of F is a somewhat unfortunate concept, as it has nothing to do with the inverse function or image of F. To make matters more confusing, it is often denoted by F^{-1}. We will avoid this by using $\widetilde{F}$ instead.

Exercise 7.4. *Let $f, g : X \to Y$ be homotopic maps, $h : W \to X$ and $k : Y \to V$ arbitrary continuous maps. Show that*

$$f \circ h \simeq g \circ h \quad \textit{and} \quad k \circ f \simeq k \circ g \ .$$

Definition 7.5 (Homotopy equivalence)**.** A map $f : X \to Y$ is called a *homotopy equivalence with homotopy inverse* g, if there exists a continuous map $g : Y \to X$ such that

$$g \circ f \simeq \mathrm{id}_X \quad \text{and} \quad f \circ g \simeq \mathrm{id}_Y \ .$$

The topological spaces X and Y are called *homotopy equivalent to each other* or to *have the same homotopy type*, if there exists a map $f : X \to Y$ which is a homotopy equivalence.

As can be seen from the definition, if g is a homotopy inverse of f, then f is a homotopy inverse of g. The homotopy inverse of a map is typically not unique, but might not exist.

Proposition 7.6. *The relation $\simeq$ is an equivalence relation on topological spaces, which is refined by $\approx$.*

Proof. The relation $\simeq$ is reflexive (take $f = \mathrm{id}$,$g = \mathrm{id}$), and symmetric (if $f : X \to Y$ is a homotopy equivalence with homotopy inverse g, then $g : Y \to X$ is a homotopy equivalence with homotopy inverse f).

To check transitivity, let $f : X \to Y$ be a homotopy equivalence with homotopy inverse $g : Y \to X$, and $h : Y \to Z$ a homotopy equivalence with homotopy inverse $k : Z \to Y$. Then

$$(g \circ k) \circ (h \circ f) \;=\; g \circ (k \circ h) \circ f \;\simeq\; g \circ \mathrm{id}_Y \circ f \;=\; g \circ f \;\simeq\; \mathrm{id}_X \ ,$$

and completely analogously for the other composition. □

The simplest spaces up to homemorphism are spaces with one point only. As there are not too many of them, this is of little help to us. Luckily, as we will see, from the point of view of homotopy, there are lots of 'interesting' spaces falling under the heading 'simplest', that is, being homotopy equivalent to a single point. This gives rise to a substantial and beautiful theory.

Definition 7.7. A topological space X is called *contractible*, if it is homotopy equivalent to the one-point space.

By unwinding this seemingly innocent definition, we arrive at the following.

Proposition 7.8. *X is contractible if and only if the identity map $\mathrm{id}_X : X \to X$ is homotopic to a map $X \to X$ whose image is a single point.*

Proof. Every continuous map $\star \to X$ is given by the image of the single point of $\star$, hence g is determined by $g(\star) \in X$. On the other hand, there is exactly one map $X \to \star$, namely, the one taking every point to $\star$. Therefore $f \circ g = \mathrm{id}_\star$, while $g \circ f : X \to X$ is the function taking every point of X to $g(\star)$.

Using this description it is easy to derive the proposition. Since $f \circ g = \mathrm{id}_\star$ anyway, X is contractible, if and only if $f \circ g \simeq \mathrm{id}_\star$. But the former is a map whose image is one point. □

Example 7.9 ($\mathbb{R}^n$ is contractible)**.** Let $X = \mathbb{R}^n$, and define $F : \mathbb{R}^n \times I \to \mathbb{R}^n$ via $F(x,t) = tx$. Then F is a homotopy from f_0, the map taking the whole space to the origin, and $f_1 = \mathrm{id}_{\mathbb{R}^n}$. Therefore $\mathbb{R}^n$ is contractible.

The previous simple example had the curious property that $f_t : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism for all positive t, while f_0 contracts the whole space.

Example 7.10 ($\mathbb{R}^n - \{0\} \simeq \mathbb{S}^{n-1}$)**.** Let $i : \mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n - \{0\}$ be the inclusion, $r : \mathbb{R}^n - \{0\} \to \mathbb{S}^{n-1}$ be the central projection

$$x \mapsto \frac{x}{\|x\|} .$$

Then $r \circ i = \mathrm{id}_{\mathbb{S}^{n-1}}$, while $i \circ r \simeq \mathrm{id}_{\mathbb{R}^n - \{0\}}$ via

$$F(x,t) \; = \; tx + (1-t)\frac{x}{\|x\|} .$$

Hence $\mathbb{R}^n - \{0\} \simeq \mathbb{S}^{n-1}$.

Challenge. One of our main goals is to develop a machinery which is capable to decide if $\mathbb{S}^1$ (or $\mathbb{S}^n$ in general) is contractible.

Definition 7.11. Let $A \subseteq X$ be an arbitrary subspace. A map $f : X \to A$ is called a *retraction*, if $f(a) = a$ for every $a \in A$. The subspace A is said to be a *retract of* X, if a retraction $f : X \to A$ exists.

Example 7.12. The map

$$\begin{array}{rcl} \pi : \mathbb{R}^2 - \{(0,0)\} & \longrightarrow & \mathbb{S}^1 \\ (x,y) & \mapsto & \left(\dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}} \right) \end{array}$$

is a retraction.

Definition 7.13. A subspace $A \subseteq X$ is called a *deformation retract of* X, if there exists a homotopy $F : X \times I \to X$ such that

$$F(x,t) \; = \; \begin{cases} x & \text{if } t = 0 \ , \\ \in A & \text{if if } t = 1 \ , \end{cases}$$

The subspace A is called a *strong deformation retract* if in addition it is required, that $F(a,t) = a$ for every $t \in I$ and $a \in A$.

In other words, A is a deformation retract, if there is a homotopy F such that $f_0 = \mathrm{id}_X$, $f_1(X) \subseteq A$; a strong deformation retract if in addition $f_t|_A = \mathrm{id}_A$ for every $t \in I$. Note that if A is a deformation retract of X, then automatically $A \simeq X$.

Definition 7.14. Let $A \subseteq X$ be an arbitrary subspace. A homotopy $F : X \times I \to Y$ is said to be *relative to* A (denoted $\mathrm{rel}\, A$) if for every $a \in A$ $F(a,t)$ is independent of t, that is, $f_t|_A$ is consant for every $t \in I$.

The next result is a complicated device, which will however make computations with homotopies quite easy.

Lemma 7.15 (Reparametrisation Lemma)**.** *Let $\phi_1, \phi_2 : (I, \partial I) \to (I, \partial I)$ be continuous maps that are equal on ∂I, $F : X \times I \to Y$ be a homotopy, $G_i(x,t) \stackrel{def}{=} F(x, \phi_i(t))$ for $i = 1, 2$. Then*

$$G_1 \simeq G_2 \ \operatorname{rel} X \times \partial I \ .$$

Proof. We define the homotopy of homotopies $H : X \times I \times I \to Y$ by

$$H(x,t,s) \stackrel{\text{def}}{=} F(x, s\phi_2(t) + (1-s)\phi_1(t)) \ .$$

Then by substituting the appropriate values in the definition of H we obtain

$$\begin{aligned} H(x,t,0) &= F(x,\phi_1(t)) \ = \ G_1(x,t) \ , \\ H(x,t,1) &= F(x,\phi_2(t)) \ = \ G_2(x,t) \ , \\ H(x,0,s) &= F(x,\phi_1(0)) \ = \ G_1(x,0) \ , \\ H(x,1,s) &= F(x,\phi_2(0)) \ = \ G_1(x,1) \ , \end{aligned}$$

with the latter two equalities coming from $\phi_1(0) = \phi_2(0)$ and $\phi_1(1) = \phi_2(1)$. ☐

Let us denote by C the constant homotopy. Note that C will depend on the context. For example, in the case of $F \star C$ C is the homotopy for which

$$C(x,t) \ = \ F(x,1)$$

for every $x \in X$, while for $C \star F$ the constant homotopy is the one with

$$C(x,t) \ = \ F(x,0) \ .$$

Proposition 7.16. *We have*

$$\begin{aligned} F \star C \ &\simeq F \ \operatorname{rel} X \times \partial I \ , \\ C \star F \ &\simeq F \ \operatorname{rel} X \times \partial I \ . \end{aligned}$$

Proof. The statements follow from the Reparametrization Lemma by letting

$$\phi_1(t) \ = \ \begin{cases} 2t & \text{if } t \leq \frac{1}{2} \\ 1 & \text{if } t \geq \frac{1}{2} \end{cases} \quad \text{and} \quad \phi_2(t) = t$$

in the first case, and

$$\phi_1(t) \ = \ \begin{cases} 0 & \text{if } t \leq \frac{1}{2} \\ 2t-1 & \text{if } t \geq \frac{1}{2} \end{cases} \quad \text{and} \quad \phi_2(t) = t$$

in the second. ☐

Proposition 7.17. *For a homotopy $F : X \times I \to Y$ one has*

$$F \star \widetilde{F} \ \simeq \ C \ \operatorname{rel} X \times \partial I \ ,$$

where $C(x,t) = F(x,0)$ for all $x \in X$ and $t \in I$.

Proof. This again follows from the Reparametrization Lemma by setting

$$\phi_1(t) \;=\; \begin{cases} 2t & \text{if } t \le \frac{1}{2} \\ 2-2t & \text{if } t \ge \frac{1}{2} \end{cases} \quad \text{and} \quad \phi_2(t) = 0 \; .$$

□

Proposition 7.18. *Let $F, G, H : X \times I \to Y$ be homotopies such that $F \star G$ and $G \star H$ are defined. Then one has*

$$(F \star G) \star H \;\simeq\; F \star (G \star H) \;\; \mathrm{rel}\, X \times \partial I \; .$$

Theorem 7.19. *Let $f : X \to Y$ be a continuous map, $\mathfrak{F}$ be the set of homotopies with $f_0 = f_1 = f$. Then $\mathfrak{F}$ is a group with multiplication and inverse given by $\star$ and $\widetilde{\cdot}$.*

8. The fundamental group and some applications

8.1. **Definition and basic properties.** The fundamental group is a very important algebraic invariant attached to a topological space. Here we will not go too deeply into the theory, we will mostly content ourselves with a few simple facts. The proof of the highly non-trivial result of computing the fundamental group of the circle $\mathbb{S}^1$ will come as a consequence of the theory of covering spaces. Nevertheless, we will use this fact to derive a number of interesting consequences.

First of all, let us remind ourselves, that a path in a topological space X is a map $f : I \to X$, where $I \stackrel{\text{def}}{=} [0,1]$. We will keep this piece of notation for the rest of these notes.

Definition 8.1. A *loop* f in a topological space X based at a point $x_0 \in X$ is a path $f : I \to X$ such that $f(0) = f(1) = x_0$

Remark 8.2. A path is nothing else than a homotopy of maps from the one-point space to X. A loop is a homotopy of maps from a map $* \to X$ to itself.

In the language of pointed spaces one can also say that a loop is a map $(\mathbb{S}^1, 1) \to (X, x_0)$.

By the previous remark, we can concatenate loops in the sense of homotopies. Note that the concatenation of any two loops based at the same point makes sense. Also, we can form the reverse homotopy just as we have seen in the case of general homotopies. As a reminder, note that if $f, g : (I, \partial I) \to (X, x_0)$ are two loops, then

$$(f * g)(s) \stackrel{\text{def}}{=} \begin{cases} f(2s) & \text{if } 0 \le s \le \frac{1}{2} \; , \\ g(2s-1) & \text{if } \frac{1}{2} \le s \le 1 \; . \end{cases}$$

On the other hand, the inverse of f is given by

$$f^{-1}(s) \stackrel{\text{def}}{=} f(1-s)$$

for every $s \in I$.

Proposition 8.3. *The set*

$$\pi_1(X, x_0) \stackrel{def}{=} \{[f] \mid f : I \to X \text{ is a loop based at } x_0\}$$

forms a group with respect to concatenation and inverse of homotopies. The identity element is the constant loop based at x_0.

Proof. This is a special case of Theorem 7.19. □

Definition 8.4. The group $\pi_1(X, x_0)$ is called the *fundamental group of X at the base-point x_0.*

Note that there is kind of a general disagreement of whether to use additive or multiplicative notation here. In any case, both 0 and 1 mean the unique group with one element.

Example 8.5. If $X \subseteq \mathbb{R}^n$ is a convex subset, then $\pi_1(X, x_0) = 1$ for every $x_0 \in X$. This follows from the observation that any two loops f_0 and f_1 based at x_0 are homotopic to each other via the linear homotopy

$$F(t, s) \;=\; (1-t)f_0(s) + tf_1(s)\ .$$

In general it is a non-trivial issue to show that certain spaces have non-trivial fundamental groups.

The next question we are going to deal with is to investigate the extent to which the group $\pi_1(X, x_0)$ depends on the base point x_0.

Remark 8.6. Since $\pi_1(X, x_0)$ involves only the path component of x_0, we can only hope to find a relation between fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x_0')$ at different points, if x_0 and x_0' lie in the same path component.

Construction 8.7. Let X be an arbitrary topological space, $x_0, x_0' \in X$ points that belong to the same path component of X. Let $h : I \to X$ be a path from x_0 to x_0', with h^{-1} denoting the inverse path of h in the sense of homotopies.

To each loop $f' : (I, \partial I) \to (X, x_0')$ we can associate the loop

$$h^{-1} * (f' * h)$$

based at x_0. This way, we establish a well-defined function

$$\begin{array}{rcl} \tau_h : \pi_1(X, x_0') & \longrightarrow & \pi_1(X, x_0) \\ [f] & \mapsto & [h^{-1} * (f' * h)]\ . \end{array}$$

Since pre- and postcomposing with maps preserves the relation of being homotopic, if f_t' is a homotopy of loops based at x_0', then $h^{-1} * (f_t' * h)$ will be a homotopy of loops based at x_0. Therefore τ_h is indeed well-defined.

Note that one has a choice between $h^{-1} * (f'h)$ and $(h^{-1} * f) * h$; although the two paths are different as maps, they are homotopic, hence end up defining the same element of $\pi_1(X, x_0)$.

Proposition 8.8. *The function of sets*

$$\tau_h : \pi_1(X, x_0') \longrightarrow \pi_1(X, x_0)$$

is an isomorphism of groups.

Proof. First of all, observe that τ_h is a homomorphism of groups, since τ_h takes a constant loop based at x_0' to a constant loop based at x_0, hence it preserves the identity, on the other hand,

$$\begin{aligned} \tau_h[f * g] &= [h^{-1} * ((f * g) * h)] \\ &= [h^{-1} * f * h * h^{-1} * g * h] \\ &= [(h^{-1} * f * h) * (h^{-1} * g * h)] \\ &= [h^{-1} * f * h] * [h^{-1} * g * h] \\ &= \tau_h[f] * \tau_h[g] \ . \end{aligned}$$

We will prove that τ_h is an isomorphism by exhibiting its two-sided inverse homomorphism. To this end, we will check that

$$\begin{aligned} \tau_{h^{-1}} \circ \tau_h &= \mathrm{id}_{\pi_1(X, x_0')} \ , \\ \tau_h \circ \tau_{h^{-1}} &= \mathrm{id}_{\pi_1(X, x_0)} \ . \end{aligned}$$

□

Corollary 8.9. *If X is a path-connected topological space, then as an abstract group, $\pi_1(X, x_0)$ is independent of the choice of x_0.*

It is very important to point out that the isomorphism between the fundamental groups at various base points is not canonical, that is, there is no natural or distinguished isomorphism between them.

Remark 8.10. The common isomorphism class of the groups $\pi_1(X, x_0)$ in case of a path-connected space is often denoted by $\pi_1(X)$, and called the *fundamental group of X*. Note that $\pi_1(X)$ is just an abstract group, it is no longer a set of homotopy classes of loops.

Definition 8.11. A topological space X is called *simply-connected*, if it is path-connected, and has $\pi_1(X) = 1$.

Proposition 8.12. *A topological space X is simply connected if and only if between any two points x_0, x_1 of X there is a unique homotopy class of paths connecting x_0 to x_1.*

Proof. Assume first that there exists a unique homotopy class of paths between any two points of X. Then, in particular, for any two points $x, y \in X$ there exists a path from x to y, consequently, X is path-connected. The set $\pi_1(X, x_0)$ consists of one element only, since by assumption there is only one homotopy class of paths from x_0 to x_0. Therefore $\pi_1(X, x_0) = 1$, and X is simply connected.

To go the other way, assume that X is simply connected. The definition includes the fact that X is path-connected as well, therefore we only need to worry about the uniqueness of the homotopy class of paths from a point x to another point y. To see this, let $f, g : I \to X$ be paths from x to y, where $x, y \in X$ are arbitrary points. Then

$$f \simeq f * (g^{-1} * g) \simeq (f * g^{-1}) * g \simeq g \ ,$$

as $f * g^{-1}$ is homotopic to the constant loop by the uniqueness of homotopy classes of paths from x to x. □

8.2. **Applications.** Let us remind ourselves to the following.

Definition 8.13. Let $A \subseteq X$ be an arbitrary subset; a *retraction of X onto A* is a continuous map $r : X \to A$ such that

$$r|_A = \mathrm{id}_A \ .$$

If such a map exists, then A is called a retract of X.

Lemma 8.14. *Let $j : A \hookrightarrow X$ denote the inclusion of A into X, $a_0 \in A$. If A is a retract of X, then the induced map*

$$j_* : \pi_1(A, a_0) \longrightarrow \pi_1(X, a_0)$$

on the fundamental groups is injective.

Proof. If $r : X \to A$ is a retraction, then the composition $r \circ j$ equals the identity of A. Therefore

$$1_{\pi_1(X,a_0)} \ = \ (\mathrm{id}_A)_* \ = \ (r \circ j)_* \ = \ r_* \circ j_*$$

by functoriality. But then j_* must be injective, since it has a left inverse. □

Theorem 8.15. *There does not exist a retraction of the 2-dimensional disk $\mathbb{D}^2$ to its boundary $\partial\mathbb{D}^2 \approx \mathbb{S}^1$.*

Proof. We argue by contradiction. Suppose there exists such a retraction $r : \mathbb{D}^2 \to \partial\mathbb{D}^2$. By Lemma 8.14 this induces an injective homomorphism

$$j_* : \pi_1(\partial\mathbb{D}^2) \hookrightarrow \pi_1(\mathbb{D}^2) \ .$$

However, $\partial\mathbb{D}^2 \approx \mathbb{S}^1$, hence its fundamental group is the infinite cyclic group, while $\mathbb{D}^2$ has trivial fundamental group, since it is convex, a contradiction. □

Definition 8.16. A map $f : X \to Y$ is *nullhomotopic*, if it is homotopic to a constant map.

Lemma 8.17. *The following statements are equivalent for any topological space X, and any continuous map $f : \mathbb{S}^1 \to X$.*

(1) f is nullhomotopic.
(2) f extends to a continuous map $\overline{f} : \mathbb{D}^2 \to X$.

(3) *The induced homomorphism*

$$f_* : \pi_1(\mathbb{S}^1, x_0) \longrightarrow \pi_1(X, f(x_0))$$

is trivial.

Proof. We will prove the direction (1) implies (2) first. Let $H : \mathbb{S}^1 \times I \to X$ be a homotopy between f and a constant map, let

$$\begin{aligned} \pi : \mathbb{S}^1 \times I &\longrightarrow \mathbb{D}^2 \\ (x,t) &\mapsto (1-t)x \ . \end{aligned}$$

Then π is continuous, closed and surjective, hence it is a quotient map. It collapses $\mathbb{S} \times \{1\}$ to $0 \in \mathbb{D}^2$, but it is otherwise injective. Since $H|_{\mathbb{S}^1 \times \{1\}}$ is continuous, it induces via π a continuous map $g : \mathbb{D}^2 \to X$ extending f.

Assume now (2), and let us prove that f_* maps every element of $\pi_1(\mathbb{S}^1, x_0)$ to the identity of $\pi_1(X, f(x_0)$. To this end, let $j : \partial\mathbb{D}^2 \hookrightarrow \mathbb{D}^2$ denote that inclusion of the boundary of $\mathbb{D}^2$. Then $f = \overline{f} \circ j$, so

$$f_* = \overline{f}_* \circ j_* \ .$$

Observe that j_* is trivial, since $\pi_1(\mathbb{D}^2)$ is. Therefore f_* has to be trivial as well.

To conclude it remains to prove that (3) implies (1). Assume accordingly that the homomorphism of groups

$$f_* : \pi_1(\mathbb{S}^1, x_0) \longrightarrow \pi_1(X, f(x_0))$$

is trivial, and consider the usual covering map $p : \mathbb{R} \to \mathbb{S}^1$ given by $s \mapsto e^{2\pi i s}$ along with its restriction $p_0 \overset{\text{def}}{=} p|_I : I \to \mathbb{S}^1$.

Then $[p_0]$ as an element of $\pi_1(\mathbb{S}^1, y)$ generates $\pi_1(\mathbb{S}^1, y)$ as it is a loop in $\mathbb{S}^1$ whose lift to $\mathbb{R}$ starting at 0 ends at 1. Let $x_0 \overset{\text{def}}{=} f(y)$. As f_* is the trivial homomorphism, the loop $k \overset{\text{def}}{=} f \circ p_0$ represents the identity of $\pi_1(X, x_0)$, therefore there exists a path homotopy F in X between k and the constant path at x_0.

The map $p_0 \times I : I \times I \to \mathbb{S}^1 \times I$ is a quotient map (one checks easily that it is continuous, closed, surjective, and injective except at $\{0\} \times I$ and $\{1\} \times I$), hence induces a continuous map $F_1 : \mathbb{S}^1 \times I \to X$, which gives a homotopy between f and the constant map. □

Theorem 8.18 (Brouwer's fixed point theorem for $\mathbb{D}^2$)**.** *If $f : \mathbb{D}^2 \to \mathbb{D}^2$ is a continuous map, then f has a fixed point.*

For the proof we will need some preparations.

Definition 8.19. A *vector field on* $\mathbb{D}^2$ is a continuous map $v : \mathbb{D}^2 \to \mathbb{R}^2$. A vector field v on $\mathbb{D}^2$ is called *non-vanishing*, if $v(x) \neq 0$ for every $x \in \mathbb{D}^2$.

Lemma 8.20. *For every vector field v on $\mathbb{D}^2$ there exist points $x, y \in \partial\mathbb{D}^2$ such that*

$$v(x) = \alpha x \ , \ v(y) = -\beta y$$

for some $\alpha, \beta > 0$ real numbers.

Proof. Suppose first that there does not exist a point $x \in \partial\mathbb{D}^2$ with $v(x)$ pointing directly inwards (ie. $v(x) = -\beta x$, $\beta > 0$). Consider the restriction

$$w \stackrel{\text{def}}{=} v\big|_{\partial\mathbb{D}^2} .$$

Because w extends to a map $\mathbb{D}^2 \to \mathbb{R}^2 - \{(0,0)\}$, it is nullhomotopic by Lemma 8.17.

On the other hand, w is homotopic to the inclusion map $j : \mathbb{S}^1 \approx \partial\mathbb{D}^2 \hookrightarrow \mathbb{R}^2 - \{(0,0)\}$ via

$$F(x,t) \stackrel{\text{def}}{=} tx + (1-t)w(x) .$$

Observe that $F(x,t) \neq 0$: this is clearly so for $t = 0, 1$; if we had $F(x,t) = 0$ for some $0 < t < 1$, then from

$$tx + (1-t)w(x) \;=\; 0$$

we could conclude $w(x) = -\frac{t}{1-t}x$, that is, $w(x)$ would point directly inwards at x, which was supposed not to be the case. Therefore F maps into $\mathbb{R}^2 - \{(0,0)\}$, and thus provides a homotopy between j and w. This implies that j is null-homotopic, contradicting the fact that it induces an isomorphism on the fundamental groups.

To show that v points directly outward at some $x \in \partial\mathbb{D}^2$, replace v by $-v$. □

Proof. (of Theorem8.18). We will argue by contradiction. Suppose that $f(x) \neq x$ for every $x \in \mathbb{D}^2$. Then

$$v(x) \stackrel{\text{def}}{=} f(x) - x$$

defines a nowhere vanishing vector field on $\mathbb{D}^2$. Observe that v cannot point directly outward at any $x \in \partial\mathbb{D}^2$, since this would imply

$$\begin{aligned} f(x) - x &= \alpha x \ (\alpha > 0) , \\ f(x) &= \alpha x \notin \mathbb{D}^2 , \end{aligned}$$

which contradicts Lemma8.20. □

Lemma 8.21. *Let $g : \mathbb{S}^1 \to \mathbb{S}^1$ be the map $g(z) = z^n$. Then*

$$g_* : \pi_1(\mathbb{S}^1, 1) \longrightarrow \pi_1(\mathbb{S}^1, 1)$$

is injective.

Proof. Consider $p_0 : I \to \mathbb{S}^1$ given by $s \mapsto e^{2\pi i s}$; it is a loop, therefore defines an element of $\pi_1(\mathbb{S}^1, 1)$. Its image under g_* is

$$g_*([p_0]) \;=\; [g \circ p_0] ,$$

where $g(p_0(t)) = e^{2\pi i n t}$ by construction. The path $g \circ p_0$ lifts to the path $t \mapsto nt$ in the covering space $\mathbb{R} \to \mathbb{S}^1$. Therefore $g_*[p_0]$ corresponds to $n \in \mathbb{Z}$ as $[p_0]$ corresponds to n.

It follows that g_* is multiplication by n, hence g_* is injective. □

Lemma 8.22. *The map $\widetilde{g} : \mathbb{S}^1 \to \mathbb{R}^2 - \{(0,0)\}$ given by $z \mapsto z^n$ is not nullhomotopic.*

Proof. If $j : \mathbb{S}^1 \hookrightarrow \mathbb{R}^2 - \{(0,0)\}$ denotes the inclusion map, then we can write

$$\widetilde{g} = j \circ g .$$

We have seen in Lemma 8.21 that the homomorphism g_* is injective. Since S^1 is a retract of $\mathbb{R}^2 - \{(0,0)\}$, j_* is injective as well by Lemma 8.14. Therefore $\widetilde{g}_*$ is also injective, hence $\widetilde{g}$ cannot be nullhomotopic. □

Theorem 8.23 (The fundamental theorem of algebra)**.** *Let $f[z] \in \mathbb{C}[z]$ be a non-constant polynomial. Then f has a root in $\mathbb{C}$.*

Proof. First we will prove the following simplified version of the theorem: if

$$f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 ,$$

where $\sum_{i=0}^{n-1} |a_i| < 1$, then f has a root in $\mathbb{D}^2 \subseteq \mathbb{C}$.

Suppose to the contrary that f has no root in $\mathbb{D}^2$, then one can define a continuous map

$$\begin{aligned} k : \mathbb{D}^2 &\longrightarrow \mathbb{R}^2 - \{(0,0)\} \\ z &\mapsto f(z) . \end{aligned}$$

Since f is supposed to have no roots in $\mathbb{D}^2$, the restriction $k|_{\mathbb{S}^1}$ extends to $\mathbb{D}^2$. Therefore $k|_{\mathbb{S}^1}$ is nullhomotopic.

On the other hand, the homotopy

$$\begin{aligned} F : \mathbb{S}^1 \times I &\longrightarrow \mathbb{R}^2 - \{(0,0)\} \\ (z,t) &\mapsto z^n + t(a_{n-1}z^{n-1} + \cdots + a_0) \end{aligned}$$

shows that the nullhomotopic $k|_{\mathbb{S}^1}$ is homotopic to $z \mapsto z^n$, which is known *not* to be homotopic to a constant map by 8.22, a contradiction.

It is important to point out that F indeed maps into $\mathbb{R}^2 - \{(0,0)\}$, since

$$\begin{aligned} |F(z,t)| &\geq |z^n| - |t(a_{n-1}z^{n-1} + \cdots + a_0)| \\ &\geq 1 - t(|a_{n-1}| + \cdots + |a_0|) \\ &> 0 . \end{aligned}$$

Hence we have verified that f has a root in $\mathbb{D}^2$ provided $\sum_{i=0}^{n-1} |a_i| < 1$.

For the general case, drop the restriction on $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$, and choose an arbitrary positive real number $c > 0$. Write $w = \frac{z}{c}$. Then

$$(cw)^n + a_{n-1}(cw)^{n-1} + \cdots + a_0 = 0$$

if and only if

$$w^n + \frac{a_{n-1}}{c}w^{n-1} + \cdots + \frac{a_0}{c^n} = 0 .$$

This latter equation however has a root in $\mathbb{D}^2$ once

$$\sum_{i=0}^{n-1} \left| \frac{a_{n-i}}{c^i} \right| < 1 .$$

□

Corollary 8.24. *Any polynomial $f \in \mathbb{C}[z]$ of degree n can be written in the form*

$$f(z) = \alpha(z - z_1) \cdot \ldots \cdot (z - z_n) \ ,$$

where $\alpha, \zeta_1, \ldots, z_n \in \mathbb{C}$, and the numbers z_i are not necessarily different.

Theorem 8.25. *If $h : \mathbb{S}^1 \to \mathbb{S}^1$ is an antipode-preserving map, then h is not null-homotopic.*

Proof. Without loss of generality we will assume that $h(1) = 1$. If this did not hold initially, then take the rotation $\rho : \mathbb{S}^1 \to \mathbb{S}^1$ mapping $h(1)$ to 1, and consider the antipode-preserving map $rho \circ h$ instead of h. If H were a homotopy between h and a constant map, then $\rho \circ H$ would provide a homotopy between $\rho \circ h$ and a constant map.

Consider the map $q : \mathbb{S}^1 \to \mathbb{S}^1$, $q(z) = z^2$. This map is continuous, closed, and surjective, hence it is a quotient map. For every $q \in \mathbb{S}^1$, one has $q^{-1}(w) = \{z, -z\}$ for suitable $z \in \mathbb{S}^1$.

Because $h(-z) = -h(z)$, one has $q(h(-z)) = q(h(z))$, therefore $q \circ h$ induces a continuous map $k : \mathbb{S}^1 \to \mathbb{S}^1$ with $k \circ q = q \circ h$:

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r, "h"] \arrow[d, "q"'] & \mathbb{S}^1 \arrow[d, "q"] \\ \mathbb{S}^1 \arrow[r, "k"] & \mathbb{S}^1 \end{tikzcd} .$$

Note that $q(1) = h(1) = 1$, so that $k(1) = 1$; in addition $h(-1) = -1$.

With this in hand, we will prove that the group homomorphism k_* is non-trivial. To this end, observe first of all, that q is a covering map. Now if $\widetilde{f}$ is a path in $\mathbb{S}^1$ from 1 to -1, then $f \overset{\text{def}}{=} q \circ \widetilde{f}$ is a loop at 1 giving a non-trivial element of $\pi_1(\mathbb{S}^1, 1)$ (since $\widetilde{f}$ is a lift starting at 1 and ending at -1, a point $\neq 1$).

Then

$$k_*[f] = [k \circ (q \circ \widetilde{f})] = [q \circ (h \circ \widetilde{f})] \ ,$$

where $h \circ \widetilde{f}$ is a path from 1 to -1, and so $q \circ (h \circ \widetilde{f})$ gives and non-trivial loop in $\mathbb{S}^1$.

Note that every non-trivial homomorphism $\mathbb{Z} \to \mathbb{Z}$ is injective, therefore k_* must be injective. Since q_* is injective as well, we obtain that $k_* \circ q_*$ is injective. But this latter equals $q_* \circ h\circ$, therefore h_* has to be injective, too. □

Theorem 8.26. *There does not exist a continuous anti-pode preserving map $g : \mathbb{S}^2 \to \mathbb{S}^1$.*

Proof. Contrary to what we need to prove, suppose that $g : \mathbb{S}^2 \to \mathbb{S}^1$ is an anti-pode preserving map. Let $E \subseteq \mathbb{S}^2$ be the equator. Since $E \approx \mathbb{S}^1$,

$$g|_E : E \to \mathbb{S}^1$$

cannot be null-homotopic by Theorem 8.25.

On the other hand, $g|E$ obviously extends to the northern hemispere, which is $\approx \mathbb{D}^2$, hence must be null-homotopic, a contradiction. □

Theorem 8.27 (Borsuk–Ulam Theorem)**.** *If $f : \mathbb{S}^2 \to \mathbb{R}^2$ is a map, then there exists $x \in \mathbb{S}^2$ for which*

$$f(x) = f(-x) \ .$$

Proof. We will again argue by contradiction. Suppose that for every $x \in \mathbb{S}^2$, one has $f(x) \neq f(-x)$. Then the function

$$\begin{aligned} g : \mathbb{S}^2 &\longrightarrow \mathbb{S}^1 \\ x &\mapsto \frac{f(x) - f(-x)}{|f(x) - f(-x)|} \end{aligned}$$

is continuous and antipode preserving. But such a map cannot exist by Theorem 8.26.

□

9. Covering spaces

In the course of the current section $p : E \to B$ denotes a surjective map of topological spaces unless otherwise mentioned.

Definition 9.1. Let $p : E \to B$ be a surjective map. An open set $U \subseteq B$ is said to be *evenly covered by* p if

$$p^{-1}(U) = \coprod_{\alpha \in I} V_\alpha$$

where for every $\alpha \in I$ the subset $V_\alpha \subseteq E$ is open, and $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism. We call the collection $\{V_\alpha \,|\, \alpha \in I\}$ the partition of $p^{-1}(U)$ into slices.

Definition 9.2. A surjective map $p : E \to B$ is called a *covering map* or a *covering space* if every $b \in B$ has an open neighbourhood $U_b \subseteq B$ which is evenly covered by p.

Remark 9.3. Note that if $p : E \to B$ is a covering map, then the fibre $p^{-1}(b) \subseteq E$ has the discrete topology for every $b \in B$. This follows from the observation that for $\alpha \in I$ we have

$$\left|V_\alpha \cap p^{-1}(b)\right| = 1 \ ,$$

therefore all points are open in $p^{-1}(b)$.

Next we show a simple but useful property of covering maps.

Proposition 9.4. *Every covering map p is open.*

Proof. As required, we will show that the image of an open set $A \subseteq E$ is open in B. To this end, pick $x \in p(A)$, and let $U \subseteq B$ be an evenly covered open neighbourhood of x with

$$p^{-1}(U) = \coprod_a V_\alpha$$

being its partition into slices. There exists $y \in A$ with $p(y) = x$, let V_β be the unique slice containing y.

Then $V_\beta \cap A \subseteq E$ is open, therefore $V_\beta \cap A \subseteq V_\beta$ is open as well. Because p maps V_β homeomorphically onto U, one has that $p(V_\beta \cap A) \subseteq U$ is an open subset as well. This implies that $p(V_\beta \cap A) \subseteq A$ is also open, hence $p(V_\beta \cap A)$ is an open neighbourhood of x contained in $p(A)$. It then follows that $p(A) \subseteq B$ is open, which is waht we wanted. $\square$

The perhaps simplest example of a covering map is the following.

Example 9.5. Let X be an arbitrary topological space, set $E \overset{\text{def}}{=} X \times \{1, 2, \ldots, n\}$, where this latter set is given the discrete topology. Then the projection

$$p : E \to X \ , \ p(x, i) = x$$

is a covering map.

To avoid such an easy way out, we will usually restrict ourselves to path-connected covering spaces. The next example is a basic one.

Proposition 9.6. *The map*

$$\begin{array}{rcl} p : \mathbb{R} & \longrightarrow & \mathbb{S}^1 \\ x & \mapsto & e^{2\pi i x} \end{array}$$

is a covering map.

One way to illustrate this map is via the composition

$$\mathbb{R} \hookrightarrow \mathbb{R}^3 \longrightarrow \mathbb{S}^1$$

where the first map takes $x \mapsto (e^{2\pi i x}, x)$, while the second one projects to the plane spanned by the first two coordinates.

Proof. To begin with, let $U \subseteq \mathbb{S}^1$ denote the subset consisting of points with positive first coordinates. Then

$$p^{-1}(U) = \{x \in \mathbb{R} \mid \cos 2\pi x > 0\} = \coprod_{n \in \mathbb{Z}} \left(n - \frac{1}{4}, n + \frac{1}{4} \right) .$$

Setting $V_n \overset{\text{def}}{=} \left(n - \frac{1}{4}, n + \frac{1}{4}\right)$, note the following.

(1) $p|_{\overline{V_n}}$ is injective since $\sin 2\pi x$ is strictly monotonously increasing on such intervals;

(2) $p|_{\overline{V_n}} : \overline{V_n} \to \overline{U}$ and $p|_{V_n} : V_n \to U$ are both surjective as a consequence of the Intermediate Value Theorem;
(3) $\overline{V_n}$ is compact and $\overline{U}$ is Hausdorff.

Putting all this together we obtain that $p|_{\overline{V_n}} : \overline{V_n} \to \overline{U}$ is a homeomorphism. Since p maps V_n bijectively onto U, the restriction $p|_{V_n} : V_n \to U$ is a homeomorphism as well. Therefore U is evenly covered by p.

Completely analogous arguments show that the subsets of $\mathbb{S}^1$ with negative first coordinates, positive second coordinates, and negative second coordinates, respectively, are all evenly covered. These four subsets however form an open cover of $\mathbb{S}^1$, therefore p is a covering map as stated. □

Definition 9.7. A map $f : X \to Y$ of topological spaces is a *local homeomorphism*, if every $x \in X$ has an open neighbourhood $U \subseteq X$, which gets mapped homeomorphically by f onto an open subset of Y.

Remark 9.8. By construction every covering map is a local homeomorphism.

Exercise 9.9. *Construct a map, which is a local homeomorphism, but not a covering map. Can you make one, which is surjective?*

Example 9.10. We can construct other covering maps of $\mathbb{S}^1$ using complex power maps: define $p_n : \mathbb{S}^1 \to \mathbb{S}^1$ by sending $z \to z^n$. In real coordinates, this map becomes

$$(\cos x, \sin x) \mapsto (\cos nx, \sin nx) \ .$$

We move on to constructing new covering spaces out of existing ones.

Proposition 9.11. *Let $p : E \to B$ be a covering map, $W \subseteq B$ an arbitrary subspace, $T \overset{def}{=} p^{-1}(W) \subseteq E$. Then $p|_T : T \to W$ is also a covering map.*

Proof. Pick $w \in W$ arbitrary, let $V \subseteq B$ be an open neighbourhood of w evenly covered by p. In particular, let

$$p^{-1}(V) = \coprod_{\alpha} V_\alpha$$

be the partition of its inverse image into slices.

Then $V \cap W$ is an open neighbourhood of $w \in W$, the sets $T \cap V_\alpha$ are disjoint and open in T, their union equals $p^{-1}(V \cap W)$; moreover, each of the $T \cap V_\alpha$'s is mapped homeomorphically onto $V \cap W$ by p. Therefore $V \cap W$ is an evenly covered open neighbourhood of w (with respect to the restriction map $p|_T$). We can conclude that $p|_T : T \to W$ is indeed a covering map. □

Next we prove a similar result for products of covering spaces.

Proposition 9.12. *If $p : E \to B$ and $p' \to E' \to B'$ are both covering spaces, then so is $p \times p' : E \times E' \to B \times B'$ defined by*

$$(e, e') \mapsto (p(e), p'(e')) \ .$$

Proof. Fix $(b, b') \in B \times B'$, and let $b \in U$, $b' \in U'$ be open neighbourhoods evenly covered by p and p', respectively. Write

$$p^{-1}(U) = \coprod_{\alpha \in I} V_\alpha \ , \ (p')^{-1}(U') = \coprod_{\alpha' \in I'} V'_{\alpha'} \ .$$

Then

$$(p \times p')^{-1}(U \times U') = \bigcup_{(\alpha,\alpha') \in I \times I'} V_\alpha \times V'_{\alpha'}$$

is a partition of $(p \times p')^{-1}(U \times U')$ into slices, since the $V_\alpha \times V'_{\alpha'}$'s are disjoint open sets in $E \times E'$, each of which is mapped homeomorphically onto $U \times U'$ under $p \times p'$. $\square$

Example 9.13 (A covering of the torus)**.** Using the covering map $p : \mathbb{R} \to \mathbb{S}^1$ constructed for the circle, we obtain the covering maps

$$p \times p : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{S}^1 \times \mathbb{S}^1 \ ,$$

which wraps the plane around the torus infinitely many times.

Let now $x_0 \stackrel{\text{def}}{=} p(0) \in \mathbb{S}^1$, and $T_0 \stackrel{\text{def}}{=} (\mathbb{S}^1 \times \{x_0\}) \cup (\{x_0\} \times \mathbb{S}^1)$ the 'figure eight' on the torus. Then the inverse image of T_0 under $p \times p$ is a covering map of T_0. This inverse image is the infinite grid $(\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R})$.

Example 9.14. Consider the composition of maps

$$p \times i : \mathbb{R} \times \mathbb{R}_+ \longrightarrow \mathbb{S}^1 \times \mathbb{R}_+ \xrightarrow{\approx} \mathbb{R}^2 \setminus \{(0,0)\}$$

given by

$$(x,t) \mapsto (e^{2\pi i x}, t) \mapsto t e^{2\pi i x} \ .$$

This gives a covering $\mathbb{R} \times \mathbb{R}_+ \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$, which is in essence the Riemann surface corresponding to the complex logarithm function.

Definition 9.15. Let $h : Y \to Z$, $f : X \to Z$ be maps of topological spaces. A *lifting of* f is a map $\widetilde{f} : X \to Y$ such that $h \circ \widetilde{f} = f$.

$$\begin{tikzcd} & Y \arrow[d, "h"] \\ X \arrow[ur, "\widetilde{f}"] \arrow[r, "f"] & Z \end{tikzcd}$$

In general liftings of maps do not exist; however, they do in many important special cases, like in the ones that will follow. These results will prove to be extremely important.

Proposition 9.16 (Lifting of paths)**.** *Let $p : E \to B$ be a covering map, $e_0 \in E$ arbitrary, $b_0 = p(e_0)$. Then any path $f : [0,1] \to B$ starting at b_0 has a unique lifting to a path $\widetilde{f}$ in E starting at e_0.*

Proof. To begin with, cover E by open sets evenly covered by p. Since the closed unit interval is compact, by the Lebesgue number lemma there exists a subdivision

$$0 = s_0 < s_1 < \ldots < s_{n-1} < s_n = 1$$

of $[0,1]$ such that $f([s_i, s_{i+1}])$ lies in an evenly covered open set for every $0 \leq i \leq n-1$.

We define $\widetilde{f}$ inductively. Let $\widetilde{f}(0) = e_0$. Assuming now that $\widetilde{f}$ is defined on a closed interval $[0, s_i]$, define $\widetilde{f}$ on $[s_i, s_{i+1}]$ as follows: if $\widetilde{f}([s_i, s_{i+1}]) \subseteq U$ for an evenly covered open subset of E, consider

$$p^{-1}(U) \;=\; \coprod_{\alpha \in I} V_\alpha \;,$$

the partition of $p^{-1}(U)$ into disjoint slices. It is a simple but important observation that $\widetilde{f}(s_i)$ lies in exactly one of the V_α's, let us denote this by V_{α_0}. Because

$$p|_{V_{\alpha_0}} : V_{\alpha_0} \longrightarrow U$$

is a homeomorphism, we can set

$$\widetilde{f}(s) \stackrel{\text{def}}{=} (p|_{V_{\alpha_0}})^{-1}(f(s)) \;,$$

which is thus uniquely defined for $s_i \leq s \leq s_{i+1}$. Also, $\widetilde{f}_{[s_i, s_{i+1}]}$ will be continuous.

Proceeding this way we define $\widetilde{f}$ on all of $[0,1]$. Clearly $\widetilde{f} : [0,1] \to E$ is continuous and $p \circ \widetilde{f} = f$ by construction.

We are left with proving that uniqueness of $\widetilde{f}$. Suppose that f' is another lifting of f beginning at the point e_0. Then of course $f'(0) = \widetilde{f}(0) = e_0$. Assume that we have

$$f'\big|_{[0,s_i]} \;=\; \widetilde{f}\big|_{[0,s_i]} \;,$$

and consider the interval $[s_i, s_{i+1}]$, and U, V_{α_0} as above, ie.

$$\widetilde{f}\big|_{[s_i,s_{i+1}]} \;=\; \Big((p|_{V_{\alpha_0}})^{-1} \circ f \Big)\big|_{[s_i,s_{i+1}]} \;.$$

Since f' is a lifting of f,

$$f'([s_i, s_{i+1}]) \;\subseteq\; p^{-1}(U) \;=\; \coprod_{\alpha \in I} V_\alpha \;.$$

The slices V_α are disjoint and $f'([s_i, s_{i+1}])$ is connected, hence it must lie entirely in one specific V_α. As $f'(s_i) = \widetilde{f}(s_i)$, this must be V_{α_0}, in other words, $f'([s_i, s_{i+1}) \subseteq V_{\alpha_0}$, in particular, for every $s \in [s_i, s_{i+1}]$, $f'(s)$ is some point of V_{α_0} lying in $p^{-1}(f(s))$. However, there is only one such point, namely

$$(p|_{V_{\alpha_0}})^{-1}(f(s)) \;=\; \widetilde{f}(s) \;.$$

From this we can conclude that

$$f'|_{[s_i,s_{i+1}]} \;=\; \widetilde{f}|_{[s_i,s_{i+1}]} \;,$$

which proves uniqueness by induction on i. $\square$

Example 9.17. Consider the covering map $p : \mathbb{R} \to \mathbb{S}^1$ with the path $f : [0,1] \to \mathbb{S}^1$ given by $f(s) = (\cos \pi s, \sin \pi s)$. Set $b_0 = (1,0)$. Then the lift $\widetilde{f}$ of f starting at $t = 0$ is the path $\widetilde{f}(s) = \frac{s}{2}$.

Proposition 9.18 (Lifting of path homotopies)**.** *Let $p : E \to B$ be a covering map, $p(e_0) = b_0$, $F : I \times I \to B$ a continuous map with $F(0,0) = b_0$. Then there is a unique lifting $\widetilde{F} : I \times I \to E$ of F such that $\widetilde{F}(0,0) = e_0$.*

Moreover, if F is a path homotopy, then so is $\widetilde{F}$.

Proof. The proof will be completely analogous to the case of paths. Given F as in the Theorem, set $\widetilde{F}(0,0) \stackrel{\text{def}}{=} e_0$.

Next, use Proposition 9.16 to extend $\widetilde{F}$ to the subsets $\{0\} \times I$ and $I \times \{0\}$. Once this is done, choose subdivisions

$$0 = s_0 < s_1 < \ldots < s_{n-1} < s_m \text{ and } 0 = t_0 < t_1 < \ldots < t_{n-1} < t_n$$

of I so that

$$I_i \times J_j \stackrel{\text{def}}{=} [s_{i-1}, s_i] \times [t_{i-1}, t_i]$$

is mapped by F into an evenly covered subset $U \subseteq B$. Just like in the case of lifting paths, this is made possible by the Lebesgue number lemma.

We will now define $\widetilde{F}$ inductively; first for the rectangles $I_1 \times J_1, I_2 \times J_1, \ldots, I_m \times J_1$, then for the $I_i \times J_2$'s and so on in lexicographic order. In general, given $1 \leq i_0 \leq m$ and $1 \leq j_0 \leq n$, assume that $\widetilde{F}$ is defined all rectangles which have an index smaller than (i_0, j_0) in the lexicographic order. Denote the union of all these by A.

Choose an evenly covered open set $U \supseteq F(I_{i_0} \times I_{j_0})$, and let

$$p^{-1}(U) = \coprod_{\alpha \in I} V_\alpha$$

as customary. By the induction hypothesis, $\widetilde{F}$ is defined on $C \stackrel{\text{def}}{=} A \cap (I_{i_0} \times J_{j_0})$, ie. on the left and bottom edges of $I_{i_0} \times J_{j_0}$. Note that C is connected, therefore so is $\widetilde{F}(C)$, hence it must lie entirely in one of the sets V_α, which we will call V_0, by connectedness, V_0 will also contain $F(I_{i_0} \times J_{j_0})$. Let

$$\widetilde{F}(s,t) \stackrel{\text{def}}{=} (p|_{V_0})^{-1}(F(s,t))$$

for $(s,t) \in I_{i_0} \times J_{j_0}$. By construction, the extended map will be continuous, and lift F.

The proof of uniqueness goes exactly the same way as in Proposition 9.16.

Suppose now that F is a homotopy of paths; we wish to show that $\widetilde{F}$ is a homotopy of paths as well (ie. it keeps the endpoints fixed). By definition, F carries $\{0\} \times I$ to a point $b_0 \in B$; now since $p \circ \widetilde{F} = F$, one has

$$\widetilde{F}(\{0\} \times I) \subseteq p^{-1}(b_0) \ .$$

Here, we have a connected subset in a discrete topological space, hence $\widetilde{F}(\{0\} \times I)$ must be just one point. A completely similar argument implies that $\widetilde{F}(\{1\} \times I)$ is equal to a point as well, so $\widetilde{F}$ is indeed a path homotopy. $\square$

Theorem 9.19. *Let $p : E \to B$ be a covering map, $p(e_0) = b_0$; consider two paths $f, g : I \to B$ from $b_0 \in B$ to $b_1 \in B$. Denote by $\widetilde{f}$ and $\widetilde{g}$ their respective liftings to E starting at e_0.*

If $f \sim g$, then $\widetilde{f}$ and $\widetilde{g}$ end at the same point, and $\widetilde{f} \sim \widetilde{g}$.

Proof. Let $F : I \times I \to B$ be a path homotopy between f and g; then $F(0,0) = b_0$. Let $\widetilde{F} : I \times I \to E$ be the lifting of F with $\widetilde{F}(0,0) = e_0$. By Proposition 9.18, $\widetilde{F}$ is a homotopy of paths, therefore

$$\widetilde{F}(\{0\} \times I) \;=\; \{e_0\} \;\;, \widetilde{F}(\{1\} \times I) \;=\; \{e_1\} \;\;,$$

where e_1 is defined by the previous equality. The bottom edge $\widetilde{F}|_{I\times\{0\}}$ is a path beginning at e_0 lifting $F|_{I\times\{0\}}$. But the lifted path is unique by Proposition 9.16, hence

$$\widetilde{F}(s,0) \;=\; \widetilde{f}(s)$$

for every $0 \le s \le 1$. In a completely analogous fashion, we obtain that $\widetilde{F}(s,1) = \widetilde{g}(s)$ on the whole interval $[0,1]$. Since we have seen earlier in the proof that $\widetilde{f}$ and $\widetilde{g}$ both end at the same point e_1, $\widetilde{F}$ is indeed a path homotopy between $\widetilde{f}$ and $\widetilde{g}$. $\square$

The following construction relates the fundamental group to the liftings of homotopies and paths we have been studying so far.

Definition 9.20 (Lifting correspondence)**.** Let $p : E \to B$ be a covering map; $b_0 \in B$, $e_0 \in p^{-1}(b_0) \subseteq E$ arbitrary points. We define a function of sets

$$\phi : \pi_1(B, b_0) \longrightarrow p^{-1}(b)$$

as follows. For an element $[f] \in \pi_1(B, b_0)$, let $\widetilde{f}$ be the lifting of f to a path in E starting at e_0; now set

$$\phi([f]) \overset{\text{def}}{=} \widetilde{f}(1) \;,$$

the endpoint of the lift $\widetilde{f}$ of f. The function ϕ is called the *lifting correspondence.*

Remark 9.21. Note that the Definition makes sense because of Theorem 9.19. Elaborate.

Also, it is easy to demonstrate by examples that ϕ actually depends on the choice of e_0.

Theorem 9.22. *With notation as above, if E is path-connected, then the lifting correspondence*

$$\phi : \pi_1(B, b_0) \longrightarrow p^{-1}(b_0)$$

is surjective. If E is simply connected, then ϕ is bijective.

Proof. Assuming E is path-connected, for any given $e_1 \in p^{-1}(b_0)$ there exists a path f' in E from e_0 to e_1. Then the composition

$$f \stackrel{\text{def}}{=} p \circ f'$$

is a loop in B at b_0; moreover $\phi([f]) = e_1$ by construction. This shows the surjectivity of ϕ.

Assume now that E is simply connected in addition, let $[f], [g] \in \pi_1(B, b_0)$ be homotopy classes of loops such that $\phi([f]) = \phi([g])$, and denote by $\widetilde{f}$ and $\widetilde{g}$ the lifts of f and g, respectively, to paths in E beginning at e_0. . Then $\widetilde{f}(1) = \widetilde{g}(1)$.

Since E is simply connected, any two paths between two points are homotopic, hence there exists a path homotopy $\widetilde{F}$ between $\widetilde{f}$ and $\widetilde{g}$. Then $F \stackrel{\text{def}}{=} p \circ \widetilde{F}$ is a path homotopy in B between f and g, therefore $[f] = [g]$ as elements of $\pi_1(B, b_0)$. □

As a consequence of the theory developed in this section, we will be able to compute the fundamental group of the circle with ease.

Theorem 9.23. $\pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$.

Proof. We will demonstrate this via the lifting correspondence associated to the covering map $p : \mathbb{R} \to \mathbb{S}^1$ given by $s \mapsto e^{2\pi i s}$.

Take $e_0 \stackrel{\text{def}}{=} 0 \in \mathbb{R}$, $b_0 = p(e_0) = (1, 0)$. Then $p^{-1}(b_0) = \mathbb{Z} \subset \mathbb{R}$. Since $\mathbb{R}$ is simply connected, the lifting correspondence

$$\phi : \pi_1(\mathbb{S}^1, b_0) \xrightarrow{\sim} \mathbb{Z}$$

is a bijective function of sets. Whence, we only need to prove that ϕ is also a homomorphism of groups, and we are done.

Let $[f], [g] \in \pi_1(\mathbb{S}^1, b_0)$ be arbitrary elements, $\widetilde{f}.\widetilde{g}$ the respective lifts of f and g to $\mathbb{R}$ starting at $e_0 = 0$. Then $n \stackrel{\text{def}}{=} \widetilde{f}(1), m \stackrel{\text{def}}{=} \widetilde{g}(1) \in \mathbb{Z}$, moreover

$$\phi([n]) = n \ , \ \phi([g]) = m \ .$$

Let furthermore g' be the path $g'(s) \stackrel{\text{def}}{=} n + \widetilde{g}(s)$ $(0 \leq s \leq 1)$. As

$$p(n + x) = p(x)$$

for every $x \in \mathbb{R}$, g' is the unique lifting of g with starting point $n \in \mathbb{R}$. It follows that the composition of paths $\widetilde{f} * g'$ is defined, and is the unique lift of $f * g$ beginning at 0. The endpoint of $\widetilde{f} * g'$ is $g'(1) = m + n$. Therefore

$$\phi([f] * [g]) = n + m = \phi([f]) + \phi([g]) \ .$$

□

Here is a grown-up version of our earlier result on the lifting correspondence.

Theorem 9.24. *Let $p : E \to B$ be a covering map, $p(e_0) = b_0$ arbitrary. Then*

(1) the induced homomorphism

$$p_* : \pi_1(E, e_0) \longrightarrow \pi_1(B, b_0)$$

is injective.

(2) Set $H \overset{def}{=} p_*(\pi_1(E, e_0))$, *then the lifting correspondence* ϕ *induces an injective map of sets*

$$\Phi : \pi_1(B, b_0)/H \longrightarrow p^{-1}(b_0) \ ,$$

where $\pi_1(B, b_0)/H$ *denotes the set of right cosets with respect to* H. *The function* Φ *is bijective provided* E *is simply-connected.*

(3) If f *is a loop in* B *based at* b_0, *then* $[f] \in H$ *precisely if* f *lifts to a* loop *in* E *based at* e_0.

Proof. (1) Let $\widetilde{h}$ be a loop in E based at e_0, suppose

$$p_*([\widetilde{h}]) \ = \ 1 \ .$$

Then there exists a path homotopy F from $p \circ \widetilde{h}$ to the constant loop in B.

If $\widetilde{F}$ is the unique lift of F in E with $\widetilde{F}(0,0) = e_0$, then by construction $\widetilde{F}$ is a path homotopy between $\widetilde{h}$ and the constant loop at e_0. Therefore $[\widetilde{h}] = 1\pi_1(E, e_0)$.

(2) Next, let f, g be loops in B with respective lifts $\widetilde{f}, \widetilde{g}$ to E starting at e_0. By construction

$$\phi([f]) \ = \ \widetilde{f}(1) \ , \ \phi([g]) \ = \ \widetilde{g}(1) \ .$$

We need to show that $\phi([f]) = \phi([g])$ holds if and only if $[f] \in H * [g]$.

Assume first that we have $[f] \in H * [g]$; then there must exist an element $[h] \in H$ for which $[f] = [h * g]$, where $h = p \circ \widetilde{h}$ for some loop $\widetilde{h}$ in E based at e_0. Hence $\widetilde{h} * \widetilde{g}$ is defined, and provides a lifting of $h * g$. As $[f] = [g * h]$, the paths $\widetilde{f}$ and $\widetilde{h} * \widetilde{g}$ must end at the same point in E, therefore so must $\widetilde{f}$ and $\widetilde{g}$. This implies $\phi([f]) = \phi([g])$.

To prove the other implication, note again that $\phi([f]) = \phi([g])$ means exactly that the paths $\widetilde{f}$ and $\widetilde{g}$ have the same endpoint. Consequently, the concatenation $\widetilde{f} * \widetilde{g}^{-1}$ is defined, and gives a loop $\widetilde{h}$ in E based at e_0 But then $[\widetilde{h} * \widetilde{g}] = [\widetilde{f}]$.

If $\widetilde{F}$ is a homotopy of paths in E between $\widetilde{h}*\widetilde{g}$ and $\widetilde{f}$, then the composition $p \circ \widetilde{F}$ is a homotopy of paths in B between $h * g$ and f with $h = p \circ \widetilde{h}$.

If E is path-connected, then ϕ is surjective, but then so is Φ.

(3) The function Φ is injective provided $\phi([f]) = \phi([g])$ and $[f] = H * [g]$ are equivalent statements. Apply this observation in the case when g is the constant loop:

$$\phi([f]) \ = \ C_{e_0} \text{ if and only if } [f] \in H \ .$$

The first of these two statements is equivalent to the claim that the lift of f to E starting at e_0 ends there as well.

□

Remark 9.25 (The monodromy representation associated to a covering map). Let $p : E \to B$ be a covering map, $b_0 \in B$ fixed, choose $[f] \in \pi_1(B, b_0)$. Then for every $e \in \pi^{-1}(b_0)$ there is a unique endpoint $\phi_e([f])$ attached to e (defined as the endpoint of the unique lift of f to E starting at e. For arbitrary elements $e, e' \in p^{-1}(b)$, we have

$$e = e' \text{ if and only if } \phi_e([f]) = \phi_{e'}([f]) .$$

by the uniqueness of lifts of paths applied to the supposed common endpoint of f and f^{-1}.

This way, $[f]$ induces a permutation σ_f of $\pi^{-1}(b_0)$ by

$$e \mapsto \phi_e([f]) .$$

Claim. With notation as above, if $p^{-1}(b_0)$ has finitely many, say d elements, then

$$\begin{array}{rcl} \rho : \pi_1(B, b_0) & \longrightarrow & \mathrm{Sym}(p^{-1}(b_0)) = S_d \\ [f] & \mapsto & \sigma_f \end{array}$$

If E is connected, then $\rho(\pi_1(B, b_0))$ is a transitive subgroup of S_d, that is, for every two permutations $e_1, e_2 \in p^{-1}(b_0)$ there exists an element $\sigma \in \operatorname{im} \rho$ for which $\sigma(e_1) = e_2$.

For more information see [4, Chapter III., Section 4].

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