Topology A (TOPA) / Alex Küronya / Spring 2009

Homework 8

Due date: May 4th

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Show that a retract of a contractible topological space is contractible.

2. ** Prove that for any pair of maps $f, g : \mathbb{S}^n \to \mathbb{S}^n$ with $\forall x \in \mathbb{S}^n\ f(x) \neq -g(x)$, one has $f \simeq g$.

3. Let $f : X \to Y$ be homotopic maps, with f being a homeomorphism. Does it follow that g is a homeomorphism as well? Prove your answer.

4. Which of the following topological properties are preserved under homotopy equivalence of topological spaces (i.e. if $X \simeq Y$ and X has the property in question then does Y have it automatically?): compactness, connectedness, path-connectedness, first countability, second countability, T_0, T_1, T_2, T_3, T_4?

5. Which of the following properties are preserved under homotopy of maps: injective, surjective, open, closed, perfect, proper?

6. * Let $\phi : G \to H$ be a map of topological groups (that is, ϕ should be a continuous homomorphism), assume that H is Hausdorff.
(i) Verify that $\ker\phi \subseteq G$ is a closed normal subgroup of G.
(ii) Prove that the factor group $G/\ker\phi$ is a topological group when equipped with the quotient topology with respect to the canonical homomorphism $G \to G/\ker\phi$.
(iii) Show that if G is compact and ϕ surjective, then

$$G/\ker\phi \approx H$$

as topological groups (in other words: prove that the map $\widetilde{\phi} : G/\ker\phi \to H$ induced by ϕ is a homeomorphism of topological spaces *and* an isomorphism of groups).

7. ** (Homotopy extension lemma) Let X be a topological space for which the product $X \times I$ is normal; let furthermore $A \subseteq X$ be a closed subspace along with a continuous map $f : A \to Y$ into an open subset Y of $\mathbb{R}^n$. Prove that if f is nullhomotopic, then it may be extended to a continuous map $g : X \to Z$ which is also homotopic.

8. * (Functorial properties of f_*) Let $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$ be continuous maps. Show that $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is a homomorphism of groups, and

$$(g \circ f)_* = g_* \circ f_* \ .$$

Moreover, verify that if i is the identity map, that it induces the identity homomorphism on the fundamental groups.

9. Prove that if $f : (X, x_0) \to (Y, y_0)$ is a homeomorphism, then

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

is an isomorphism.

10. ** Show that if X is a path-connected topological space, $h : X \to Y$ a continuous map, then h_* is independent of the base points chosen (up to appropriate isomrphisms). (Note: Your first task is to write it down precisely what it means for f_* to be independent of base points.)

11. ** Prove that for any topological group G, the fundamental group $\pi_1(G, 1)$ is abelian.

12. Decide if the following statements are true:
   (1) If X is a discrete topological space with n elements (n a positive integer), $x_0 \in X$, then $\pi_1(X, x_0)$ has exactly n elements.
   (2) The fundamental group of a trivial topological space at an arbitrary base-point is trivial.