HOMEWORK 7

Due date: April 24$^{\text{th}}$

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. * Prove that if $f : X \to Y$, $g : Z \to W$ are open identification maps, then so is $f \times g : X \times Z \to Y \times W$, where $(f \times g)(x, z) \overset{\text{def}}{=} (f(x), g(z))$.

Definition. Let X,Y be topological spaces, $A \subseteq X$ a closed subset, $f : A \to Y$ a map. We define

$$Y \cup_f X \overset{\text{def}}{=} \left(X \coprod Y\right) /\!\sim \ ,$$

where $\sim$ is the equivalence relation generated by the set $\{(a, f(a) \mid a \in A\}$. The procedure is called *attaching X to Y along f.*

2. With notation as in the previous definition, show that for arbitrary points $u, v \in X \cup Y$, one has $u \sim v$ if and only if one of the following conditions hold (i) $u = v$; (ii) $u, v \in A$ and $f(u) = f(v)$; (iii) $u \in A$, $v \in Y$, and $f(u) = v$.

3. Let X, Y be topological space, $A \subseteq X$ a closed subset, $f : A \to Y$ a map. Consider the quotient space $Y \cup_f X$. Show that the natural inclusion $Y \hookrightarrow Y \cup_f X$ maps Y onto a closed subspace, while the image of the inclusion $X - A \hookrightarrow Y \cup_f X$ is open.

4. Let X, Y be normal topological spaces, $A \subseteq X$ a closed subset with a closed map $f : A \to Y$. Verify that $Y \cup_f X$ is normal as well. (** Show that the statement holds even if we remove the hypothesis on f)

5. * Let G be a topological group, $H \subseteq G$ a closed subgroup. Prove that the space

$$G/H \overset{\text{def}}{=} \{ gH \mid g \in G \}$$

of left cosets of H equipped with the quotient topology induced by the canonical projection $\pi : G \to G/H$ is a Hausdorff topological space.

6. Verify the following claims for a topological group G:
   (1) The diagonal map $\delta : G \to G \times G$, $\delta(x) \overset{\text{def}}{=} (x, x)$ is closed.
   (2) $\{1\} \subseteq G$ is a closed subgroup.
   (3) The intersection of all neighbourhoods of 1 is $\{1\}$.

7. For a topological group G, show that every open subgroup of G is closed, and every closed subgroup of finite index in G is open.

8. Prove that if G is a topological group, H a subgroup of G, then G/H is discrete if and only if $H \subseteq G$ is open.

9. Show that the product of two quotient maps need not be a quotient map.

Definition. If $A \subseteq X$ is an arbitrary subspace, $\sim$ an equivalence relation on X, then the *saturation of A* is defined as

$$\operatorname{sat}(A) \overset{\text{def}}{=} \{x \in X \mid \exists a \in A : x \sim a\} \ .$$

10. ** Let $\sim$ be an equivalence relation on X, $A \subseteq X$ a subspace such that A intersects every equivalence class of $\sim$ non-trivially. Prove that there exists a well-defined continuous function

$$k : A/\!\sim \longrightarrow X/\!\sim \ ,$$

which is a homeomorphism provided that saturation of every open set in A is open in X.

11. Let (X, τ) be a topological space, and consider the equivalence relation $x \sim y$ if for every $U \in \tau$, $x \in U$ if and only if $y \in U$. Is it true that $X/\sim$ is a T_0-space?

12. Prove that $\mathbb{R}^2 \approx \mathbb{R}^1 \times \mathbb{R}^1$.