Homework 5

Due date: April 3rd

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Let (X, d) be a metric space, $A \subseteq X$ an arbitrary fixed subset. For $x \in X$ define

$$d(x, A) \stackrel{\text{def}}{=} \inf_{y \in A} d(x, y) \ .$$

Show that the function $x \mapsto d(x, A)$ is continuous.

2. ** (Lebesgue number lemma) Let (X, d) be a compact metric space, $\mathfrak{U}$ an open covering of X. Prove that there exists a positive real number δ (depending on $\mathfrak{U}$) such that for every $A \subseteq X$ with diameter less than δ, there is an element $U \in \mathfrak{U}$ for which $A \subseteq U$. (Note: the diameter of the subset A is is defined as $\sup\{d(x, y) \mid \forall x, y \in A\}$.)

3. Let (X, d_X),(Y, d_Y) be metric spaces with X compact, and let $f : X \to Y$ be a continuous function. Then f is *uniformly continuous*, that is, for every $\epsilon > 0$ there exists $\delta > 0$ such that for every pair of points $x_1, x_2 \in X$

$$d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \epsilon \ .$$

4. * Prove that if X,Y are connected topological spaces then so is $X \times Y$.

5. Let X, Y, Z be topological spaces, $f : X \times Y \to Z$ an arbitrary function. Is it true that f is continuous if and only if it is continuous in both variables separately?

Definition. A *topological group* G is a topological space equipped with a group structure in such a way that

(1) the multiplication map $\mu : G \times G \to G$, $(g, h) \mapsto gh$ is continuous,
(2) taking inverse images $i : G \to G$, $g \mapsto g^1$ is continuous.

A *subgroup* of a topological group is an abstract subgroup with the subspace topology. If G and H are topological groups then a function $f : G \to H$ is a *homomorphism of topological groups*, if it is a homomorphism of abstract groups, which is continuous.

6. Let G be a group, which is a topological space at the same time. Show that G is a topological group iff the function $G \times G \to G$ sending (x, y) to xy^{-1} is continuous.

7. Verify that the following groups (equipped with the classical topology) are topological groups: $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}^+, \cdot)$, (complex numbers of absolute value 1, $\cdot$).

8. ** Prove that the general linear group $\mathrm{GL}(n, \mathbb{R})$ together with the usual matrix multiplication and forming inverse matrices is a topological group (here $\mathrm{GL}(n, \mathbb{R}) \subseteq \mathrm{Mat}_n(\mathbb{R})$ denotes the set of $n \times n$ invertible matrices; we give this group the topology inherited from $\mathrm{Mat}_n(\mathbb{R})$ thought of as $\mathbb{R}^{n^2}$).

9. Let G be a topological group, and denote the connected component of G containing the identity by G°. Show that G° is a closed normal subgroup of G.

10. Let (X, τ) and (X, σ) be topological spaces on the same set X, assume that $\sigma \subseteq \tau$. Does the compactness of (X, τ) imply that of (X, σ)? What about the other way around?

11. Show that a finite union of compact subspaces of a topological space X is again compact.

12. * Show the following strengthening of the T_4 property. Let (X, τ) be a normal topological space, $F, G \subseteq X$ a pair of disjoint closed subsets. Then there exist open sets $U, V \subseteq X$ for which $F \subseteq U, G \subseteq V$, and $\overline{U} \cap \overline{V} = \emptyset$.