Topology (TOP) / Alex Küronya / Spring 2006

Homework 9

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. ** (Homotopy extension lemma) Let X be a topological space for which the product $X \times I$ is normal; let furthermore $A \subseteq X$ be a closed subspace along with a continuous map $f : A \to Y$ into an open subset Y of $\mathbb{R}^n$. Prove that if f is nullhomotopic, then it may be extended to a continuous map $g : X \to Z$ which is also homotopic.

2. * (Functorial properties of f_*) Let $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$ be continuous maps. Show that

$$(g \circ f)_* = g_* \circ f_* \ ,$$

moreover, verify that if i is the identity map, that it induces the identity homomorphism on the fundamental groups.

3. Prove that if $f : (X, x_0) \to (Y, y_0)$ is a homeomorphism, then

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

is an isomorphism.

4. ** Show that if X is a path-connected topological space, $h : X \to Y$ a continuous map, then h_* is independent of the base points chosen (up to appropriate isomrphisms). (Note: Your first task is to write it down precisely what it means for f_* to be independent of base points.)

5. ** For any topological group G, the fundamental group $\pi_1(X, 1)$ is abelian.

6. If $p : X \to Y$ and $q : Y \to Z$ are covering maps, and $(q \circ p)^{-1}(z)$ is finite for every $z \in Z$, then $q \circ p$ is also a covering map.

7. * Using the method for determining the fundamental group of the circle, prove that

$$\pi_1(T^2) \simeq \mathbb{Z} \times \mathbb{Z} \ .$$

8. Let $p : E \to B$ be a covering map with E path-connected. Show that if B is simply connected, then p is a homeomorphism.