Topology (TOP) / Alex Küronya / Spring 2006

Homework 8

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Show that the retract of a contractible topological space is contractible.

2. ** Prove that for any pair of maps $f, g : \mathbb{S}^n \to \mathbb{S}^n$ with $\forall x \in \mathbb{S}^n\ f(x) \neq -g(x)$, one has $f \simeq g$.

3. Let X,Y be topological spaces, Y discrete. Verify that the projection map $\pi : X \times Y \to X$ is a covering map.

4. * Consider a covering map $p : E \to B$ with B connected. Prove that if for some $x \in B$ the fibre $p^{-1}(x)$ has m elements, then $p^{-1}(b)$ has m elements for every $b \in B$. (The covering map is then called an *m-fold covering of B*.)

5. Let $f : E \to B$ be a covering map. Show that if B is Hausdorff/regular then so is E.

6. ** Let $\phi : G \to H$ be a map of topological groups (that is, a continuous homomorphism).
(i) Verify that $\ker\phi \subseteq G$ is a closed normal subgroup of G.
(ii) Show that if G is compact, then

$$G/\ker\phi \approx H$$

as topological groups (in other words: show that the map $\tilde{\phi} : G/\ker\phi \to H$ induced by ϕ is a homeomorphism of topological spaces and an isomorphism of groups).

7. Let $f, g : X \to Y$ be homotopic maps with f being a homeomorphism. Does it follow that g is a homeomorphism as well? Prove your answer.

Definition. For a topological space X, the *suspension of* X is defined to be the quotient of $X \times I$ obtained by collapsing the subspace $X \times \{0\}$ to one point, and $X \times \{1\}$ to another point. The suspension of X is denoted by $S(X)$. Let $f : X \to Y$ be a map of topological spaces. Then the *suspension of* f is the map $S(f) : S(X) \to S(Y)$ induced by

$$f \times \mathbf{1}_I : X \times I \to Y \times I\ .$$

8. Prove that $S(\mathbb{S}^n) \approx \mathbb{S}^{n+1}$.

9. Decide if the following is true: if $f, g : X \to Y$ are two homotopic maps, then $S(f)$ and $S(g)$ are homotopic as well.

10. * Show that if a topological space X deformation retracts to a point $x \in X$, then for each neighbourhood U of x there exists a neighbourhood $x \in V \subseteq U$ such that the inclusion map $V \hookrightarrow U$ is nullhomotopic.