Topology (TOP) / Alex Küronya / Spring 2006

Homework 6

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. We call a subset X in a topological group G *symmetric*, if $X^{-1} = X$. Show that the symmetric neighbourhoods of the identity element 1_G of G form a neighbourhood basis of 1_G.

2. * (Tube lemma) Let X be an arbitrary, Y a compact topological space, $x_0 \in X$ an arbitrary point, $N \subseteq X \times Y$ an open subset containing $\{ x_0 \} \times Y$. Prove that there exists an open neighbourhood of W of x_0 in X such that $N \supseteq W \times Y$.

3. Consider a set X, and two topologies τ, τ' on X. Show that if both (X, τ) and (X, τ') are compact Hausdorff, then τ and τ' are not comparable (as subsets of 2^X).

4. * Take two disjoints compact subspaces $F, G \subseteq X$, where X is Hausdorff. Prove that there exist disjoint open sets $U, V \subseteq X$ for which $F \subseteq U$ and $G \subseteq V$.

5. ** Let X be a non-empty compact Hausdorff space with no isolated points. Show that X must be uncountable.

6. Prove that a connected metric space is either uncountable or has at most one point.

Definition. Let X be a topological space, $x \in X$. We say that X is *locally compact at* x, if there exists a compact subset $C \subseteq X$ which containts a neighbourhood of x. The space X is called *locally compact* if it is locally compact at every one of its points.

7. Prove that $\mathbb{R}^n$ is locally compact, but $\mathbb{Q}$ is not.

8. Let X be a second countable topological space, $A \subseteq X$ an uncountable set. Verify that uncountably many point of A are limit points of A.

9. Prove that if X has a countable dense subset, then every collection of disjoint open sets is countable.

10. ** Let $\phi : A \to B$ a homomorphism of commutative rings (all rings in this course have 1).
   (1) Prove that the function $\tilde{\phi} : \operatorname{Spec} B \to \operatorname{Spec} A$
   $$\tilde{\phi}(P) \stackrel{\text{def}}{=} \phi^{-1}(P)$$
   is continuous.
   (2) If ϕ is surjective then $\tilde{\phi}$ is a homeomorphism of $\operatorname{Spec} B$ onto the closed subset $V(\ker \phi) \subseteq \operatorname{Spec} A$.
   (3) If ϕ is injective, then $\tilde{\phi}(\operatorname{Spec} B) \subseteq \operatorname{Spec} A$ is dense.

11. ** Let A be a commutative Boolean ring, i.e. one where $a^2 = a$ for every element $a \in A$. Prove that $\operatorname{Spec} A$ is a compact Hausdorff space.