Topology (TOP) / Spring 2006 / Alex Küronya

Homework 1

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Come up with a definition for convergence and Cauchy sequences in metric spaces.

2. (*i*) Show that the functions $s, p : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$\begin{aligned} s(x,y) &= x+y \\ p(x,y) &= xy \end{aligned}$$

are continuous.
(*ii*) Let $f, g : X \longrightarrow \mathbb{R}$ continuous functions. Then all of $f \pm g, f \cdot g$ are continuous; if $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is continuous as well.

3. (*i*) Is the function $f : \mathbb{R}^2 - \{(0,0)\} \longrightarrow \mathbb{R}^2$

$$f(x,y) = \left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2} \right)$$

continuous on $\mathbb{R}^2 - \{(0,0)\}$?
(*ii*) Is there a continuous function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ for which

$$g|_{\mathbb{R}^2-\{(0,0)\}} = f \ ?$$

4. Let $f : X \longrightarrow Y$ be a homeomorphism, x_k a sequence in X. Then x_k is convergent in X if and only if $f(x_k)$ is convergent in Y.

5. * Let α, β, γ be arbitrary real numbers. Then the so-called *open half-space*

$$H = \big\{ (x,y,z) \in \mathbb{R}^3 \,|\, \alpha x + \beta y + \gamma z > 0 \big\}$$

is indeed open.

6. Let $X \subseteq \mathbb{R}^n$, $p \in X$, $\delta > 0$. Then $\mathbb{B}_X(p, \delta)$ is open in X.

7. Prove that the set

$$\big\{ (x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \geq 10 \big\}$$

is closed.

8. Is the set consisting of all point of the form $\frac{1}{n}$, n a natural number, open/closed in $\mathbb{R}$?

9. Give examples of infinitely many open sets in $\mathbb{R}$, the intersection of which is (*i*) open (*ii*) closed (*iii*) neither open nor closed.

10. Show that the closed ball

$$D(x.\delta) = \{ y \in \mathbb{R}^n \,|\, |x-y| \leq \delta \}$$

is indeed a closed subset of $\mathbb{R}^n$.

11. * Prove that

$$d_1(f,g) = \int_{[a,b]} |f-g| dx$$

is a metric on $\mathcal{C}[a,b]$. Is this still true if we replace continuous functions by Riemann integrable ones?