Introduction to Algebraic Geometry (IAG) / Fall 2014 / Alex Küronya

Homework 9

Due date: December 8th

> The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Let $P \in C \stackrel{\text{def}}{=} V(f) \subseteq \mathbb{A}^2_{\mathbb{C}}$ be a smooth point. We say that P is a *flex* of C, if the tangent line ℓ of C at P satisfies $\operatorname{ord}^f_P(\ell) \geq 3$.

(1) Let $C \stackrel{\text{def}}{=} V(y - x^n)$ where n is a positive integer. For which values of n is $P = (0,0)$ a flex of C?
(2) Let $C \stackrel{\text{def}}{=} V(y + ax^2 +$ higher degree terms$)$. Show that $(0,0)$ is a flex of C if and only if $a = 0$.

2. Let $(R, \mathfrak{m})$ be a discrete valuation ring with quotient field K. Prove that if $\alpha \in K \setminus R$, then $\alpha^{-1} \in \mathfrak{m}$.

3. Let $p \in \mathbb{Z}$ be a prime number, and set

$$\mathbb{Z}_p \stackrel{\text{def}}{=} \{\alpha \in \mathbb{Q} \mid \alpha = \frac{a}{b},\ a \text{ and } b \text{ integers such that } p \not| b\} .$$

(1) Show that $\mathbb{Z}_p$ is a discrete valuation ring with quotient field $\mathbb{Q}$.
(2) Verify that every discrete valuation ring with quotient field $\mathbb{Q}$ is one of the above.

4. Let K be an arbitrary field. An *order function on* K is a function $\phi\colon K \to \mathbb{Z} \cup \{\infty\}$ satisfying the following properties. For evey $a, b \in K$ we have

(1) $\phi(a) = \infty$ if and only if $a = 0$,
(2) $\phi(ab) = \phi(a) + \phi(b)$,
(3) $\phi(a + b) \geq \min\{\phi(a), \phi(b)\}$.

Verify the following claims.

(1) The set $R \stackrel{\text{def}}{=} \{\alpha \in K \mid \phi(\alpha) \geq 0\}$ is a DVR with maximal ideal $\mathfrak{m} = \{\alpha \in K \mid \phi(\alpha) > 0\}$ and quotient field K.
(2) For a given DVR $(R, \mathfrak{m})$ with quotient field K, the associated function $\operatorname{ord}_R\colon K \to \mathbb{Z} \cup \{\infty\}$ is an order function in the above sense.

5. Let k be an arbitrary field. Determine

$$\dim_k k[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^r$$

as a function of n and r.

6. For an arbitrary field k, prove that the ring of formal power series in one variable $k[[x]]$ is a discrete valuation ring with uniformizing parameter x.

7. Let $(R, \mathfrak{m})$ be a DVR, ord be the associated order function on the quotient field K. Check the following claims.

(1) For every $\alpha, \beta \in K$, $\operatorname{ord}(\alpha) < \operatorname{ord}(\beta)$ implies $\operatorname{ord}(\alpha + \beta) = \beta$.
(2) If $\alpha_1, \ldots, \alpha_r \in K$, and there exists $1 \leq i \leq r$ with $\operatorname{ord}(\alpha_i) < \operatorname{ord}(\alpha_j)$ for all $j \neq i$, then

$$\sum_{i=1}^{r} \alpha_i \neq 0 .$$

8. Let R be an arbitrary ring, $\alpha = \sum_{i=0}^{\infty} a_i t^i \in R[[t]]$ a formal power series. Verify that α is a unit in $R[[t]]$ if and only if a_0 is a unit in R.

9. Let R be an integral domain with quotient field K. We call R *a valuation ring of* K, if for every element $\alpha \in K^{\times}$, we have either at least $\alpha \in R$ or $\alpha^{-1} \in R$. Now let R be a valution ring of K as defined above.

(1) Prove that R is local.
(2) Check that whenever R' is an integral domain with $R \subseteq R' \subseteq K$, then R' is a valuation ring of K as well.

10. $* + *$ Let k be an arbitrary field, $C \subseteq \mathbb{A}^2_k$ an irreducible affine variety, $P \in V$ a point. Verify the following statements for the local ring $\mathcal{O}_{C,P}$.

(1) Let $t \in \mathcal{O}_{C,P}$ be a uniformizing parameter, $\alpha \in R$ arbitrary. Then for every $n \in \mathbb{N}$ there exist uniquely defined $a_0, \ldots, a_n \in k$ and $\alpha_n \in \mathcal{O}_{C,P}$ such that

$$\alpha = a_0 + a_1 t + \cdots + a_n t^n + \alpha_n t^{n+1} .$$

(2) The previous construction yields an injective k-algebra homomorphism

$$\begin{aligned} \iota \colon \mathcal{O}_{C,P} &\longrightarrow k[[x]] \\ \alpha &\mapsto \sum_{i=0}^{\infty} a_i x^i \end{aligned}$$

that we call the power series expansion of α in terms of the uniformizing parameter t.

(3) The above homomorphism extends to an injective k-algebra homomorphism $\mathcal{O}_{C,P} \to k((x))$, where the latter denotes the quotient field of the integral domain $k[[x]]$.

(4) Let C be the affine line, and P be the origin. Choose a uniformizing parameter t, and use it to compute the associated power series expansion of the element $\frac{1}{t-1}$.