Introduction to Algebraic Geometry (IAG) / Fall 2014 / Alex Küronya

Homework 8

Due date: November 24th

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Let k be an arbitrary field. Prove that

$$\dim_k k[x,y]/(x,y)^r = \binom{r+1}{2} .$$

2. Let k be an algebraically closed field, $I \lhd k[x_1,\ldots,x_n]$ an ideal with $V(I) = \{P_1,\ldots,P_m\}$ a finite set. Set

$$\mathcal{O}_i \stackrel{\text{def}}{=} \mathcal{O}_{\mathbb{A}^n,P_i} .$$

Prove the following.

(1) $k[x_1,\ldots,x_n]/I \simeq \prod_{i=1}^m \mathcal{O}_i/I\mathcal{O}_i$.
(2) $\dim_k k[x_1,\ldots,x_n]/I = \sum_{i=1}^m \dim_k \mathcal{O}_i/I\mathcal{O}_i$.
(3) If $V(I)$ consists of a single point P, then

$$k[x_1,\ldots,x_n]/I \simeq \mathcal{O}_{\mathbb{A}^n,P}/I\mathcal{O}_{\mathbb{A}^n,P} .$$

3. Let M be an abelian group. Verify that an R-module structure on M is exactly the same as a ring homomorphism $R \to \mathrm{Hom}_{\mathbb{Z}}(M,M)$.

4. Let $0 \to V_1 \to \ldots \to V_n \to 0$ be an exact sequence of vector spaces over k. Check that

$$\sum_{i=1}^n (-1)^i \dim_k V_i = 0 .$$

5. Let M, M' be R-modules. Show that the set of all R-homomorphisms $\mathrm{Hom}_R(M,M')$ carries a natural R-module structure.

6. * Let M be an R-module, $N \leq M$.

(1) Check that the R-module structure on M/N given by $r \cdot \overline{m} \stackrel{\text{def}}{=} \overline{rm}$ is well-defined.
(2) Show that the function $\pi\colon M \to M/N$ is a surjective R-homomorphism.
(3) (Universal property of quotient modules) With notation as above, assume that $\phi\colon M \to M'$ is an R-module homomorphism satisfying $\phi(N) = 0$. Prove that there exists a unique R-homomorphism $\overline{\phi}\colon M/N \to M'$ for which $\overline{\phi} \circ \pi = \phi$.

7. * (Noether's second isomorphism theorem) Let $P \leq N \leq M$ be R-modules. Prove that there exist natural (not depending on choices) R-homomorphisms $M/P \to M/N$ and $N/P \to M/P$ for which the sequence

$$0 \longrightarrow N/P \longrightarrow M/P \longrightarrow M/N \longrightarrow 0$$

is exact.

8. Let k be an arbitrary field, $P \in C \subseteq \mathbb{A}^2_k$ a plane curve, $\mathfrak{m} \lhd \mathcal{O}_{C,P}$ be the unique maximal ideal. Prove that

$$\dim_k \mathcal{O}_{C,P}/\mathfrak{m}^r < \infty$$

for all $r \geq 1$.