Introduction to Algebraic Geometry (IAG) / Fall 2014 / Alex Küronya

Homework 7

Due date: November 17th

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. * Let V and W be irreducible affine varieties over an algebraically closed field. Show that V and W are birationally equivalent if and only if there exist non-empty open subsets $U \subseteq V$ and $T \subseteq W$ for which $U \simeq T$.

2. Let R be a ring, $X \stackrel{\text{def}}{=} \operatorname{Spec} R$ as a topological space, $f, g \in R$. Denote

$$X_f \stackrel{\text{def}}{=} \operatorname{Spec} R \setminus V(f) \ .$$

Check that following statements.

(1) The subsets $X_f \subseteq X$ are open in the Zariski topology.
(2) The collection $\{X_f \mid f \in R\}$ forms a basis for the Zariski topology on $\operatorname{Spec} R$ (that is, every open subset of X can be written as a union of sets of the form X_f).
(3) $X_f \cap X_g = X_{fg}$.
(4) $X_f = \emptyset$ precisely if f is nilpotent.
(5) $X_f = X$ if and only if f is a unit in R.
(6) $X_f = X_g$ exactly if $\sqrt{(f)} = \sqrt{(g)}$.
(7) Every open cover of X_f has a finite subcover.

3. Let $\phi \colon R \to S$ be a homomorphism of rings, $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$ as topological spaces. Prove the following claims.

(1) If $P \lhd S$ is prime ideal, then so is $\phi^{-1}(P) \subseteq R$. Therefore

$$\begin{array}{rcl} \phi^* \colon Y & \longrightarrow & X \\ P & \mapsto & \phi^{-1}(P) \end{array}$$

gives a well-defined function.
(2) If $f \in R$, then $(\phi^*)^{-1}(X_f) = Y_{\phi(f)}$; in particular, ϕ^* is continuous with respect to the Zariski topology.
(3) If $J \lhd R$, then $(\phi^*)^{-1}(V(I)) = V((\phi(I)))$.
(4) If $J \lhd S$, then $\overline{\phi^*(V(J))} = V(\phi^{-1}(J))$.
(5) If ϕ is surjective, then ϕ^* is a homeomorphism of Y onto the closed subset $V(\ker \phi) \subseteq X$.
(6) $\operatorname{Spec} R$ and $\operatorname{Spec}(R/\operatorname{Nil}(R))$ are naturally homeomorphic.
(7) If ϕ is injective, then $\phi^*(Y) \subseteq X$ is dense.
(8) $\phi^*(Y) \subseteq X$ is dense if and only if $\ker \phi \subseteq \operatorname{Nil}(R)$.
(9) If $\psi \colon S \to T$ is a homomorphism of rings, then we have $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

4. * Let $V \subseteq \mathbb{A}^n_{\mathbb{C}}$ be an irreducible affine variety, $P \in V$. Show that there is a bijective correspondence between prime ideals in $\mathcal{O}_{V,P}$, and the irreducible subvarieties of V passing through P.

5. Let k be a field, and let $k[[x]]$ denote the ring of formal power series with coefficients in k. Verify that $k[[x]]$ is a local ring.

6. Let $k = \mathbb{R}$ or $\mathbb{C}$, and let $k\{x\}$ denote the ring of power series with appropriate coefficients that are convergent around 0. Check that $k\{x\}$ is also a local ring.

7. Let $\phi \colon V \to W$ be a regular map between irreducible affine varieties, let $P \in V$, and $Q = \phi(P) \in W$.

(1) Show that $\phi^* \colon k[W] \to k[V]$ extends to a k-algebra homomorphism $\phi^* \colon \mathcal{O}_{W,Q} \to \mathcal{O}_{V,P}$.
(2) Check that $\phi^*(\mathfrak{m}_{W,Q}) \subseteq \mathfrak{m}_{V,P}$.
(3) Decide if ϕ^* always extends to $k(W)$ as well.

8. Show that a principal ideal domain (PID) is a unique factorization domain (UFD). If you have difficulties with the proof, look it up in a standard algebra textbook.

9. Let R be a UFD, K its quotient field. Check that every element $\alpha \in K$ can be written uniquely (up to multiplication by units) in the form $\alpha = a/b$, where $a, b \in R$ have no common factors.

10. Let $R = \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}},\infty} \stackrel{\text{def}}{=} \{\frac{f}{g} \in \mathbb{C}(\mathbb{A}^1) \mid \deg f \leq \deg g\}$. Show that R is a DVR with uniformizing parameter $\frac{1}{x}$.