Homework 6

Due date: October 27$^{\text{th}}$

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Decide if the following statements are true or false.
   (1) A finite union of nowhere dense subsets is again nowhere dense.
   (2) A finite intersection of dense subsets is again dense.
   (3) Let X be an irreducible noetherian topological space, $U \subseteq X$ a non-emtpy open subset. If $B \subseteq U$ is dense, then so is $B \subseteq X$.
   (4) The image of an irreducible subvariety of $\mathbb{A}^n_{\mathbb{C}}$ under a regular map is irreducible.
   (5) The image of an irreducible subvariety of $\mathbb{A}^n_{\mathbb{C}}$ under a regular map is irreducible subvariety of the target.

2. Let R be a ring where for every element $a \in R$ there exists a positive integer $n_a > 2$ such that $a^{n_a} = a$. Prove that every prime ideal in R is maximal.

3. (The spectrum of a ring) Let R be a ring, and let $\operatorname{Spec} R$ denote the set of its prime ideals. For an arbitrary subset $S \subseteq R$, define

$$V(S) \overset{\text{def}}{=} \{P \in \operatorname{Spec} R \mid S \subseteq P\} .$$

Prove the following claims.
   (1) $V(\{0\}) = \operatorname{Spec} R$, and $V(\{r\}) = \emptyset$ whenever $r \in R^\times$.
   (2) $V(S) = V((S)) = V(\sqrt{(S)})$ (where (S) denotes the ideal generated by S in R).
   (3) If $\{S_i \mid i \in I\}$ is an arbitrary collection of subsets of R, then

$$V(\cup_{i\in I} S_i) = \bigcap_{i\in I} V(S_i) .$$

   (4) If $I, J \lhd R$, then $V(IJ) = V(I \cap J) = V(I) \cup V(J)$.
   (5) The collection $\{V(S) \mid S \subseteq R\}$ determines a topology on $\operatorname{Spec} R$, the *Zariski topology.*
   (6) Describe the Zariski topology on $\mathbb{Z}$, $\mathbb{R}[x]$, $\mathbb{C}[x]$. Which points are closed?

3. Let X be an arbitrary topological space, $A, B \subseteq X$ subsets.
   (1) Show that $\operatorname{int}(A) = \{\, x \in X \mid \exists U \subseteq X \text{ open for which } x \in U \subseteq A \,\}$.
   (2) Prove that $\overline{A} = \{\, x \in X \mid \forall U \subseteq X \text{ open with } x \in U : U \cap A \neq \emptyset \,\}$.
   (3) Show that A is open if and only if $A = \operatorname{int}(A)$, and A is closed if and only in $\overline{A} = A$. Verify furthermore that $X - \operatorname{int}(A) = \overline{X - A}$, and $X - \overline{A} = \operatorname{int}(X - A)$.
   (4) Prove the following identities: $\operatorname{int}(A) \cap \operatorname{int}(B) = \operatorname{int}(A \cap B)$, $\overline{A} \cup \overline{B} = \overline{A \cup B}$.
   (5) Verify that

$$\bigcap_{\alpha\in I} \operatorname{int}(A_\alpha) \supseteq \operatorname{int}\left(\bigcap_{\alpha\in I} A_\alpha\right) = \operatorname{int}\left(\bigcap_{\alpha\in I} \operatorname{int}(A_\alpha)\right) , \quad \bigcup_{\alpha\in I} \operatorname{int}(A_\alpha) \subseteq \operatorname{int}\left(\bigcup_{\alpha\in I} A_\alpha\right) .$$

   (6) Show that $A \subseteq B$ implies $\operatorname{int}(A) \subseteq \operatorname{int}(B)$, and $\overline{A} \subseteq \overline{B}$.

4. Let $V \subseteq \mathbb{A}^n_{\mathbb{C}}$ be a non-empty affine variety, $\phi \in \mathbb{C}[V]$. Show that $V_V(\phi) = \emptyset$ if and only if ϕ is invertible in $\mathbb{C}[V]$. What happens if we ask the same question over $\mathbb{R}$?

5. Let X, Y be affine variety. What is the relationship between the Zariski topology on $X \times Y$ and the product topology?

6. Let X be an arbitrary topological space. Show that X is Hausdorff precisely if $\Delta(X) \subseteq X \times X$ is a closed subset.

6. * Let X be an irreducible affine variety over an algebraically closed field, $U \subseteq X$ a non-empty open subset. We say that a rational function $\phi \in k(X)$ is *regular on* U, if it is defined at every point $P \in U$, the k-algebra of all such functions is denoted by $\mathcal{O}_X(U)$.

(1) Show that $\mathcal{O}_X(U) = \mathcal{O}_{X,P}$ as subsets of $k(X)$.

(2) Prove that $\phi \in k(X)$ is regular on U if and only if for every $P \in U$ there exists an open subset $P \in V \subseteq U$ and polynomials $f, g \in k[x_1, \ldots, x_n]$ such that

(a) $g(Q) \neq 0$ for all $Q \in V$,

(b) $\phi(Q) = f(Q)/g(Q)$ for all $Q \in V$.

7. * Determine $\mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}(\mathbb{A}^2_{\mathbb{C}} \setminus \{(0,0)\})$.

8. ** Let X be an irreducible affine variety. Look up the definition of a sheaf of rings on a topological space and show that the function

$$U \subseteq X \text{ open} \quad \mapsto \quad \mathcal{O}_X(U)$$

defines a sheaf of rings on the topological space (X, τ_{Zar}).