HOMEWORK 5

Due date: October 20$^{\text{th}}$

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Decide if the following statements are true or false.
   (1) Let X be a topological space, $Y, Z \subseteq X$ irreducible subspaces. Then $Y \cap Z$ is irreducible as well.
   (2) The affine variety corresponding to $\mathbb{C}[X,Y]/(XY)$ is irreducible.
   (3) Every open subset of $\mathbb{A}^2$ in the Zariski topology is dense.
   (4) The affine variety corresponding to $\mathbb{C}[X,Y](XY-1)$ is irreducible.

2. Verify that every prime ideal P of a ring R can be obtained as the kernel of a homomorphism to a field.

3. Show that every non-empty open subset in an irreducible topological space is dense.

4. Identify the field k with the affine line $\mathbb{A}^1_k$ and show that regular functions $f\colon X \to k$ from algebraic sets are continuous in the Zariski topology. Does the same statement hold for an arbitrary regular map between affine varieties?

5. Give an example of a regular map $f\colon \mathbb{A}^1_k \to \mathbb{A}^2_k$ which is *not* an isomorphism, but it is a homeomorphism onto its image (with respect to the Zariski topology).

6. In which points of the circle $V(x^2+y^2-1) \subseteq \mathbb{A}^2$ is the rational function $\phi = \frac{y-1}{x}$ regular?

7. * In which points of the curve $X = V(y^2 - x^3 - x^2) \subseteq \mathbb{A}^2$ is the rational function y/x regular? Show that $y/x \notin k[X]$, but some power of it is regular.

8. Verify the following claims about dimensions of noetherian topological spaces; let X be a noetherian topological space.
   (1) If $Y \subseteq X$ is an arbitrary subspace, then Y is noetherian as well; furthermore $\dim Y \leq \dim X$.
   (2) Let $\{U_i \mid i \in I\}$ be an open cover of X. Then
   $$\dim X \;=\; \sup_{i \in I} \dim U_i \;.$$
   (3) If $U \subseteq X$ is a dense open subset, then $\dim U = \dim X$.
   (4) Let X be a finite-dimensional irreducible space, $Y \subseteq X$ a closed subspace. Then $\dim Y = \dim X$ implies $Y = X$.

Definition. An ideal $I \lhd k[x_1,\ldots,x_n]$ is called *zero-dimensional* if
$$\dim_k k[x_1,\ldots,x_n]/I < \infty \;.$$

9. Prove that an ideal I is zero-dimensional if and only if $V(I)$ is a finite set.

10. Let k be an arbitrary field, $\{p_1,\ldots,p_m\} \subseteq \mathbb{A}^n_k$ a finite set. Construct a polynomial f such that
$$f(p_i) = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } i > 1 \;. \end{cases}$$

11. Show that
$$\dim_{\mathbb{C}} \mathbb{C}[x_1,\ldots,x_n]/I \;\leq\; \#V(I)$$
for every zero-dimensional ideal I, with equality if and only if I is radical.

12. Let $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be a zero-dimensional ideal, $f \in \mathbb{C}[x_1, \ldots, x_n]$. Let $\mu_f : \mathbb{C}[x_1, \ldots, x_n]/I \to \mathbb{C}[x_1, \ldots, x_n]/I$ denote multiplication by f as a linear map.

(1) Check that μ_f is indeed a linear map. Show that $\mu_f = \mu_g$ as linear maps if and only if $f - g \in I$.
(2) Prove that $\mu_{f+g} = \mu_f + \mu_g$ and $\mu_{fg} = \mu_f \cdot \mu_g$ for every $g \in \mathbb{C}[x_1, \ldots, x_n]$.
(3) Verify that $\mu_{h(f)} = h(\mu_f)$ for every polynomial $h \in \mathbb{C}[t]$.

13. With notation as above, prove that the eigenvalues of μ_f coincide with the values of the function $f\colon V(I) \to \mathbb{C}$.

14. Decompose the affine variety $V(x^2 - yz, xz - x) \subseteq \mathbb{A}^3_{\mathbb{C}}$ into irreducible components.

15. * Let $X, Y \subseteq \mathbb{A}^n$ be affine varieties, $\Delta \stackrel{\text{def}}{=} V(x_1 - y_1, \ldots, x_n - y_n) \subseteq \mathbb{A}^{2n}$, and check that

$$\phi : X \cap Y \to (X \times Y) \cap \Delta \ , \ x \mapsto (x, x)$$

is an isomorphism.

16. Determine the image of the regular map $f : \mathbb{A}^2 \to \mathbb{A}^2$, $f(x, y) \stackrel{\text{def}}{=} (x, xy)$, and describe it from the point of view of topology.