Introduction to Algebraic Geometry (IAG) / Fall 2014 / Alex Küronya

Homework 4

Due date: October 13$^{\text{th}}$

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. (Homogeneous polynomials) Let k be an arbitrary field, $f \in k[x_1,\dots,x_n]$, d a positive integer. We say that f is *homogeneous of degree* d, if for every $\lambda \in k$ we have

$$f(\lambda x_1,\dots,\lambda x_n) = \lambda^d \cdot f(x_1,\dots,x_n) \ .$$

A homogeneous polynomial of degree d is often referred to as a *form of degree* d. Prove the following statements regarding homogeneous polynomials.

(1) A polynomial f is homogeneous of degree d if and only if it is a k-linear combination of monomials of degree d.
(2) Every polynomial f can be written uniquely as a sum of homogeneous polynomials. These are called the *homogeneous components of* f.
(3) * (Euler's theorem) Assume that k has characteristic zero. Then a polynomial $f \in k[x_1,\dots,x_n]$ is homogeneous of degree d if and only if

$$\sum_{i=1}^{n} x_i \cdot \partial_{x_i} f = d \cdot f \ .$$

2. Prove the following about colon ideals. Let $I, J, K \lhd R$.

(1) $(I \colon R) = R$.
(2) $IJ \subseteq K$ precisely if $I \subseteq (K \colon J)$.
(3) $(J \colon I)$ if and only if $(I \colon J) = R$.
(4) Let $\{I_\alpha \mid \alpha \in A\}$ a collection of ideals in R. Then

$$(I \colon \sum_{\alpha\in A} I_\alpha) = \bigcap_{\alpha \in A} (I \colon I_\alpha) \ .$$

(5) $((I \colon J) \colon K) = (I \colon JK)$.

3. Show that if $f \in k[x_1,\dots,x_n]$ is an irreducible polynomial, then $V(f) \subseteq \mathbb{A}^n_k$ is an irreducible affine variety.

4. Let $I \lhd k[x_1,\dots,x_n]$ be a proper ideal, where k is algebraically closed. Show that

$$\sqrt{I} = \bigcap_M M \ ,$$

where M runs through all maximal ideals containing I.

5. Let k be an infinite field. Check that any linear subspace of $\mathbb{A}^n_k$ is irreducible.

6. * Let $V \subseteq W \subseteq \mathbb{A}^n_k$ be affine varieties. Show that every irreducible component of V is contained in an irreducible component of W.

7. Let R be an arbitrary ring, $\{P_1,\dots,P_r\}$ a set of prime ideals. Verify that $P_1 \cap \dots \cap P_r$ is prime if and only if there exists $1 \le i \le r$ such that P_i is contained in all other P_j's.

8. Let k be algebraically closed, $f \in k[x_1,\dots,x_n]$ such that

$$f = f_1^{\alpha_1} \cdot \ldots \cdot f_r^{\alpha_r}$$

is its decomposition into irreducible factors. Check that

$$V(f) = V(f_1) \cup \dots \cup V(f_r)$$

is the irreducible decomposition of $V(f)$, and that $I(V(f)) = (f_1,\dots,f_r)$.

9. Let $V(f)$ be a plane curve of degree d over an algebraically closed field having a point of multiplicity d. Prove that $V(f)$ consists of d distinct lines.

10. Find the singular points of the following complex plane curves and describe all tangent lines at them.
   (1) $x^d - y^d$
   (2) $(x^2+y^2)^2 + 3x^2y - y^3$
   (3) $(x^2+y^2)^3 - 4x^2y^2$
   (4) $y^3 - y^2 + x^3 - x^2 + 3xy^2 + 3x^2y + 2xy$
   (5) $x^4 + y^4 - x^2y^2$

11. ** Let R be a ring, I an ideal that is maximal with respect to not being finitely generated. Show that I must be prime.

12. ** Let R be a ring, I an ideal which is maximal among the ones that are not principal. Prove that I is necessarily prime.

13. Let $I_1, \ldots, I_m$ be ideals in R, P a prime ideal in R that contains $\cap_{i=1}^m I_i$.
   (1) Show that there exists an index $1 \le i \le m$ such that $P \supseteq I_i$;
   (2) if furthermore $P = \cap_{i=1}^m I_i$, then $P = I_i$ for some $1 \le i \le m$.

14. ** Let $C \subseteq \mathbb{A}^2_{\mathbb{C}}$ be an irreducible plane curve. Show that C has only finitely many singular points.