INTRODUCTION TO ALGEBRAIC GEOMETRY (IAG) / FALL 2014 / ALEX KÜRONYA

HOMEWORK 3

Due date: October 6$^{\text{th}}$

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. If I and J are ideals in R, then show that $(I\colon J) = (I\colon \sqrt{J})$. Does $(I\colon J) = (\sqrt{I}\colon J)$ hold as well?

2. Determine the Zariski closure of the following subsets of affine space.
   (1) The first quadrant in $\mathbb{A}^2_{\mathbb{R}}$.
   (2) The set of all points with rational coordinates on the x-axis in $\mathbb{A}^2_{\mathbb{R}}$.
   (3) The set of all points with rational coordinates on the x-axis in $\mathbb{A}^1_{\mathbb{R}}$.
   (4) The set of all points with zero imaginary part and irrational real part in $\mathbb{A}^1_{\mathbb{C}}$.
   (5) The set of all points with integral coordinates in $\mathbb{A}^3_{\mathbb{C}}$.
   (6) The set of all points with integral coordinates in $\mathbb{A}^3_{\mathbb{Q}}$.
   (7) The interior of the unit disc in $\mathbb{A}^2_{\mathbb{R}}$.
   (8) The graph of the sine function inside $\mathbb{A}^2_{\mathbb{R}}$.

3. Let $I \lhd R$, and let

$$\begin{aligned} \pi\colon R &\longrightarrow R/I \\ r &\mapsto r+I \end{aligned}$$

denote the canonical surjection. Prove that the association

$$\begin{aligned} \text{Ideals in } R/I &\longrightarrow \text{Ideals in } R \text{ containing } I \\ J &\mapsto \pi^{-1}(J) \end{aligned}$$

is bijective and inclusion-preserving.

4. Let $\phi\colon R \to S$ be a homomorphism of rings. Show that
   (1) $\ker\phi \stackrel{\text{def}}{=} \{r \in R \mid \phi(r) = 0\}$ is an ideal in R, moreover all ideals of R are of this form for suitable ϕ.
   (2) $\operatorname{im}\phi \stackrel{\text{def}}{=} \{\phi(r) \mid r \in R\}$ is a subring of S, but not necessarily an ideal.
   (3) The homomorphism ϕ gives rise to an isomorphism (bijective morphism)

$$\begin{aligned} \overline{\phi}\colon R/I &\longrightarrow \operatorname{im}\phi \\ r+I &\mapsto \phi(r)\ . \end{aligned}$$

5. Verify that the set of nilpotent elements in a ring R forms an ideal (which we denote by $\operatorname{Nil}(R)$ and call the *nilradical* of R). Check that $R/\operatorname{Nil}(R)$ contains no non-zero nilpotent elements.

6. Show that $\sqrt{I}$ equals the intersection of all prime ideals in R containing I.

7. The *Jacobson radical* $\operatorname{Jac}(R)$ of a ring R is the intersection of all of its maximal ideals. Check that for an element $r \in R$, $r \in \operatorname{Jac}(R)$ if and only if $1 - rs \in R^\times$ for all $s \in R$.

8. Check that the sum of a nilpotent element and a unit is always a unit.

9. * Show that a polynomial $a_0 + a_1x + \ldots + a_nx^n \in R[x]$ is a unit if and only if $a_0 \in R^\times$, and all other coefficients are nilpotent.

10. Prove that a polynomial $a_0 + a_1x \ldots + a_nx^n \in k[x]$ (as always k denotes a field) is nilpotent, if and only if all of its coefficients are nilpotent.

11. Let (X, τ) be a topological space, $A \subseteq X$ an arbitrary subset. We define that *subspace topology* on A with respect to τ (denoted by $(A, \tau_{\mathbb{A}})$) via

$$B \in \tau_A \ \text{ if and only if there exists } U \in \tau \text{ for which } U \cap A = B.$$

Show that (A, τ_A) is indeed a topological space, and that τ_A is the smallest topology on A with respect to which the natural inclusion $\iota\colon A \hookrightarrow X$ is continuous.