Introduction to Algebraic Geometry (IAG) / Fall 2014 / Alex Küronya

Homework 2

Due date: September 29$^{\text{th}}$

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Let $I \lhd R$ be an ideal generated by one element $r \in R$. Check that $I = \{ar \mid a \in R\}$.

2. Let $V, W \subseteq \mathbb{A}^n_k$ be affine varieties. Show that
   (1) $V \subseteq W$ if and only if $I(V) \supseteq I(W)$;
   (2) $V = W$ if and only if $I(V) = I(W)$.

3. Verify that $\sqrt{I}$ is a radical ideal for all ideals $I \lhd R$.

4. Determine all maximal, prime, and radical ideals in $\mathbb{Z}$, $\mathbb{C}[x]$, and in $\mathbb{R}[x]$.

5. Let $I \lhd R$ be an arbitrary ideal. Show that
   (1) I is maximal if and only if R/I is a field.
   (2) I is prime if and only if R/I is an integral domain.

6. A ring element $r \in R$ is called *nilpotent*, if there exists a positive integer n with $r^n = 0$. Prove that $r \in R$ is nilpotent if and only if it is contained in all prime ideals.

7. Verify the following equalities between ideals.
   (1) $(2x^2 + 3y^2 - 11, x^2 - y^2 - 3) = (x^2 - 4, y^2 - 1)$
   (2) $(x + xy, y + xy, x^2, y^2) = (x, y)$.

8. * Let k be an infinite field, $C \overset{\text{def}}{=} \{(t, t^3, t^4) \mid t \in \} \subseteq \mathbb{A}^3_k$. Show that C is an algebraic variety, and determine its vanishing ideal.

9. Prove that an algebraically closed field is always infinite.

10. Let $I = (x^2 + y^2 - 1, y - 1) \subseteq \mathbb{R}[x, y]$. Give an example for $f \in I(V(I)) \setminus I$.

11. * Let k be an arbitrary field; prove that $\sqrt{(x^m, y^n)} = (x, y)$ for all positive integers m and n.

Definition. *Let $I, J \subseteq R$ be two ideals. Then the* sum $I + J$ *is defined to be*

$$I + J \overset{def}{=} \{r + s \mid r \in I, s \in J\} .$$

12. Generalize the sum of ideals for an arbitrary set of them, and prove the following statements for sums of ideals in a ring R.
    (1) The sum of an arbitrary set of ideals is indeed an ideal in R.
    (2) The sum of ideals is the smallest ideal containing all of them.
    (3) If $\{I_\alpha \mid \alpha \in A\}$ is an arbitrary collection of ideals, then $V(\sum_{\alpha \in A} I_\alpha) = \cap_{\alpha \in A} V(I_\alpha)$.

13. Show that we have $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$ for all ideals $I, J \lhd R$.

14. Let $I, J \subseteq k[x_1, \ldots, x_n]$ be two ideals. Check that $V(I \cap J) = V(I) \cup V(J) = V(IJ)$.

15. Verify for the following claims for radicals of ideals.
    (1) If $I^m \subseteq J$ for some natural number $m > 0$, then $\sqrt{I} \subseteq \sqrt{J}$.
    (2) $\sqrt{I + J} = \sqrt{\sqrt{I} + \sqrt{J}}$.
    (3) $\sqrt{IJ} = \sqrt{I \cap J}$, but $\sqrt{IJ} \neq \sqrt{I}\sqrt{J}$ in general. Show by example that the product of radical ideals is not a radical ideal in general.

16. We say that two ideals $I, J \subseteq R$ are *coprime* or *comaximal*, if $I + J = R$. Show that

(1) if I and J are comaximal, then $IJ = I \cap J$.
(2) if $R = \mathbb{C}[x_1, \ldots, x_n]$, then I and J are comaximal if and only if $V(I) \cap V(J) = \emptyset$.

17. An ideal $I \lhd k[x_1, \ldots, x_n]$ is called *monomial*, if it has a generating set consisting of monomials. Show that if I is a monomial ideal, then the monomials contained in I form a k-vector space basis of I.