Introduction to Algebraic Geometry (IAG) / Fall 2014 / Alex Küronya

Homework 10

Due date: March 15th

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Let $f, g \in \mathbb{C}[x,y]$, $P = (0,0) \in \mathbb{A}^2_{\mathbb{C}}$. Compute the intersection number

$$i(f \cap g; P) \overset{\text{def}}{=} \dim \mathcal{O}_{\mathbb{A}^2,P}/(f,g)$$

for all pairs f and g from the following set:

- $y - x^2$
- $y - x^3$
- $y^2 - x^3 + x^2$
- $y^2 - x^3$
- $(x^2+y^2)^2 + 3x^2y - y^3$
- $(x^2+y^2)^3 - 4x^2y^2$
- $x^4 + y^4 - x^2y^2$

2. * With notation as above, if $P \in C = V(f)$ is a simple point, then

$$i(f \cap g; P) = \operatorname{ord}_P^f(g) .$$

3. With notation as above, if $V(f)$ and $V(g)$ have no common components, then

$$\sum_{P \in V(f) \cap V(g)} i(f \cap g; P) = \dim_{\mathbb{C}} \mathbb{C}[x,y]/(f,g) .$$

4. (5-Lemma) Consider the following diagram with exact rows:

$$\begin{tikzcd}
A \arrow[r, "\phi_1"] \arrow[d, "\alpha"] & B \arrow[r, "\phi_2"] \arrow[d, "\beta"] & C \arrow[r, "\phi_3"] \arrow[d, "\gamma"] & D \arrow[r, "\phi_4"] \arrow[d, "\delta"] & E \arrow[d] \\
A' \arrow[r, "\psi_1"] & B' \arrow[r, "\psi_2"] & C' \arrow[r, "\psi_3"] & D' \arrow[r, "\psi_4"] & E'
\end{tikzcd}$$

Check the following claims:

(1) * If β and δ are injective, α surjective, then γ is injective.
(2) If β and δ are surjective, ϵ injective, then γ is surjective.
(3) If α,β, δ, and ϵ isomorphisms, then so is γ.