INTRODUCTION TO ALGEBRAIC GEOMETRY (IAG) / FALL 2014 / ALEX KÜRONYA

HOMEWORK 1

Due date: September 22nd

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Let $p \in \mathbb{Z}$ be a prime number, and let $\mathbb{F}_p$ denote the set of equivalence classes of $\mathbb{Z}$ modulo the prime p with the usual addition and multiplication. Check that $\mathbb{F}_p$ is a field.

2. Let R be an integral domain. Verify that the polynomial ring $R[x_1, \ldots, x_n]$ is always an integral domain, but never a field.

3. (Division with remainder) Let k be a field, $g \neq 0 \in k[x]$. Show for every $f \in k[x]$ can be written in the form

$$f = qg + r$$

with $q, r \in k[x]$, and either $r = 0$ or $\deg r < \deg g$; in addition r and q are unique.

4. Verify that every singleton set $\{a\} \in \mathbb{A}^n_k$ is an affine variety. Show that if k is a finite field, then every subset of $\mathbb{A}^n_k$ is an affine variety.

5. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$. Show that f is the zero polynomial whenever f vanishes at every point with integral coordinates.

6. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$, and let d be the largest x_i-degree of f for $0 \leq i \leq n$. Prove that f is the zero polynomial, if $f(a_1, \ldots, a_n) = 0$ for all points $(a_1, \ldots, a_n) \in \mathbb{A}^n_{\mathbb{C}}$ with $1 \leq a_i \leq d+1$ for all $1 \leq i \leq n$.

7.* Let $S \stackrel{\text{def}}{=} \{(a, a) \mid a \in \mathbb{C}^\times\}$. Decide if $S \subseteq \mathbb{A}^2_{\mathbb{C}}$ is an affine variety. Answer the same question over the field $\mathbb{F}_{17}$.

8. Consider the set $\mathbb{Z}^n \subseteq \mathbb{A}^n_{\mathbb{C}}$ consisting of points with only integer coordinates. Is this an affine variety?

9. Decide if the following statements are true or false.
   (1) An arbitrary union of affine varieties is an affine varieties.
   (2) The complement of an affine variety is again an affine variety.
   (3) The set-theoretic difference of two affine varieties is an affine variety.

10.* Let $V \subseteq \mathbb{A}^n_k$ and $W \subseteq \mathbb{A}^m_k$ be affine varieties; we define their cartesian product as

$$V \times W \stackrel{\text{def}}{=} \{(a, b) \in \mathbb{A}^{n+m}_l \mid a \in V, b \in W\} \subseteq \mathbb{A}^{n+m}_k .$$

Show that $V \times W \subseteq \mathbb{A}^{n+m}_k$ is an affine variety.

11.** Let k be a field that is *not* algebraically closed, and let $X = V(f_1, \ldots, f_m) \subseteq \mathbb{A}^n_k$ be an affine variety defined by finitely many equations [1]. Prove that X can be given as the zero locus of a single polynomial.

12. Describe the zero loci of the following ideals:
   (1) $(xy, xz, yz) \subseteq k[x, y, z]$,
   (2) the ideal generated by all square-free monomials of degree d in the variables $x_1, \ldots, x_n$ over a field k, where $1 \leq d \leq n$.

[1] We will soon see that all affine varieties satisfy this condition thanks to Hilbert's Basis Theorem.