Topology (TOP) / Alex Küronya / Fall 2013

Homework 8

Due date: Dec 2nd

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Show that if $f : X \to Y$ is an surjective map which is open or closed, then f is a quotient map.

2. Prove that the product of two open surjective maps is an open surjective map as well.

3. * Let (G, μ_G, i_G) and (H, μ_H, i_H) be topological groups. Show that $G \times H$ with the product topology and the product operations is again a topological group.

4. ** Show that the product of two quotient maps need not be a quotient map.

Definition. Let X,Y be topological spaces, $A \subseteq X$ a closed subset, $f : A \to Y$ a map. We define

$$Y \cup_f X \stackrel{\text{def}}{=} \left(X \coprod Y\right) /\sim ,$$

where $\sim$ is the equivalence relation generated by the set $\{(a, f(a) \mid a \in A\}$. The procedure is called *attaching X to Y along f*.

5. With notation as in the previous definition, show that for arbitrary points $u, v \in X \cup Y$, one has $u \sim v$ if and only if one of the following conditions hold (i) $u = v$; (ii) $u, v \in A$ and $f(u) = f(v)$; (iii) $u \in A$, $v \in Y$, and $f(u) = v$.

6. Let X, Y be topological space, $A \subseteq X$ a closed subset, $f : A \to Y$ a map. Consider the quotient space $Y \cup_f X$. Show that the natural inclusion $Y \hookrightarrow Y \cup_f X$ maps Y onto a closed subspace, while the image of the inclusion $X - A \hookrightarrow Y \cup_f X$ is open.

7. Let X, Y be normal topological spaces, $A \subseteq X$ a closed subset with a closed map $f : A \to Y$. Verify that $Y \cup_f X$ is normal as well. (** Show that the statement holds even if we remove the hypothesis on f)

8. For a topological group G, show that every open subgroup of G is closed, and every closed subgroup of finite index in G is open.

9. Prove that if G is a topological group, H a subgroup of G, then G/H is discrete if and only if $H \subseteq G$ is open.

11. * Let (X, τ) be a topological space, and consider the equivalence relation $x \sim y$ if for every $U \in \tau$, $x \in U$ if and only if $y \in U$. Is it true that $X/\sim$ is a T_0-space?