Topology (TOP) / Alex Küronya / Fall 2013

Homework 7

Due date: November 25th

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. * Show that the product of two Hausdorff topological spaces is again Hausdorff.

2. Decide which of the following statements hold.
   (1) The product of two T_0 topological spaces is again T_0.
   (2) The product of two T_1 topological spaces is T_1 as well.
   (3) The product of a T_0 space and a T_1 spaces is T_0.
   (4) The product of a T_0 space and a T_1 spaces is T_1.

Definition. A *topological group* G is a topological space equipped with a group structure in such a way that
   (1) the multiplication map $\mu : G \times G \to G$, $(g, h) \mapsto gh$ is continuous,
   (2) taking inverse images $i : G \to G$, $g \mapsto g^1$ is continuous.

If G and H are topological groups then a function $f : G \to H$ is a *homomorphism of topological groups*, if it is a homomorphism of abstract groups, which is continuous.

3. Let G be a group, which is a topological space at the same time. Show that G is a topological group iff the function $G \times G \to G$ sending (x, y) to xy^{-1} is continuous.

4. Verify that the following groups (equipped with the classical topology) are topological groups: $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}^+, \cdot)$, (complex numbers of absolute value 1, $\cdot$).

5. Let (G, τ) be a topological group with product μ and inversion ι, $H \leqslant G$ an arbitrary subgroup (in the sense of algebra). Show that $(H, \tau|_H)$ with the appropriate restrictions of μ and ι is again a topological group.

6. Let $X = \mathbb{Z}$, τ be the topology generated by $\{[n, +\infty) \mid n \in \mathbb{Z}\}$. Show that $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is continuous, but $- : \mathbb{Z} \to \mathbb{Z}$ is not.

7. Let G be a topological group, $g \in G$. Show that the function

$$\begin{aligned} L_g : G &\longrightarrow G \\ h &\mapsto gh \end{aligned}$$

(called *left translation by g*) is continuous. One can analogously define the right translation maps $R_g : G \to G$ by $h \mapsto hg$. Show that

$$L_g \circ L_h = L_{gh} \ , \ L_g \circ R_h = R_h \circ L_g \ , \ R_g \circ R_h = R_{hg}$$

for every $g, h \in G$.

8. * Prove that the general linear group $\mathrm{GL}(n, \mathbb{R})$ together with the usual matrix multiplication and forming inverse matrices is a topological group (here $\mathrm{GL}(n, \mathbb{R}) \subseteq \mathrm{Mat}_n(\mathbb{R})$ denotes the set of $n \times n$ invertible matrices; we give this group the topology inherited from $\mathrm{Mat}_n(\mathbb{R})$ thought of as $\mathbb{R}^{n^2}$).

9. We call a subset X in a topological group G *symmetric*, if $X^{-1} = X$. Show that the symmetric neighbourhoods of the identity element 1_G of G form a neighbourhood basis of 1_G.

Definition. A topological space X is called *homogeneous*, if for every pair of points $x, y \in X$ there exists a homeomorphism $\phi_{x,y} : X \to X$ for which $\phi_{x,y}(x) = y$.

10. Verify that a topological group is homogeneous as a topological space.