Topology A (TOPA) / Alex Küronya / Fall 2013

Homework 5

Due date: November 11th

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Let X be a compact topological space, $\{A_\alpha \mid \alpha \in I\}$ an arbitrary collection of closed sets, which is closed with respect to finite intersections. If for an open set $U \subseteq X$ one has $\cap_\alpha A_\alpha \subseteq U$, then there exists $\alpha \in I$ for which $A_\alpha \subseteq U$.

2. Let (X, τ) and (X, σ) be topological spaces on the same set X, assume that $\sigma \subseteq \tau$. Does the compactness of (X, τ) imply that of (X, σ)? What about the other way around?

3. Show that a finite union of compact subspaces of a topological space X is again compact.

4. Prove that if $f : X \to Y$ is a continuous map of topological spaces, X compact, Y Hausdorff, then f is closed.

5. ** Let (X, d) be a metric space, $f : X \to X$ a contraction, that is, a continuous function for which there exists a positive real number $c < 1$ such that for every pair of points $x, y \in X$,

$$d(f(x), f(y)) \leq c \cdot d(x, y) .$$

Show that if X is compact, then f has a unique fixed point (i.e. an element $x \in X$, for which $f(x) = x$).

6. ** Show that for any commutative ring R (with unit), $\operatorname{Spec} R$ is compact in the Zariski topology.

7. ** (Lebesgue number lemma) Let (X, d) be a compact metric space, $\mathfrak{U}$ an open covering of X. Prove that there exists a positive real number δ (depending on $\mathfrak{U}$) such that for every $A \subseteq X$ with diameter less than δ, there is an element $U \in \mathfrak{U}$ for which $A \subseteq U$. (Note: the diameter of the subset A is is defined as $\sup \{d(x, y) \mid \forall x, y \in A\}$.)

8. Show the following strengthening of the T_4 property. Let (X, τ) be a normal topological space, $F, G \subseteq X$ a pair of disjoint closed subsets. Then there exist open sets $U, V \subseteq X$ for which $F \subseteq U, G \subseteq V$, and $\overline{U} \cap \overline{V} = \emptyset$.

9. Prove that a noetherian topological space is compact.

10. Verify that a Hausdorff noetherian topological space must be a finite set with the discrete topology.

11. Describe all connected subsets of the real line.

12. Consider a set X, and two topologies τ, τ' on X. Show that if both (X, τ) and (X, τ') are both compact Hausdorff, then τ and τ' are not comparable (as subsets of 2^X).

13. * Take two disjoints compact subspaces $F, G \subseteq X$, where X is Hausdorff. Prove that there exist disjoint open sets $U, V \subseteq X$ for which $F \subseteq U$ and $G \subseteq V$.

14. Give $\mathbb{R}$ the topology consisting of all subsets with a countable complement and $\emptyset$. Is $[0, 1]$ compact with respect to the subspace topology?

15. * Show that a topological space X is Hausdorff if and only if for every topological space Y and any two maps $f, g \colon Y \to X$ the set

$$\{y \in Y \mid f(y) = g(y)\} \subseteq Y$$

is closed.