Topology (TOP) / Alex Küronya / Fall 2013

Homework 3

Due date: October 21st

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

Definition. A topological space X is said to be *irreducible* if whenever $X = F \cup G$, where F, G are closed, then either $X = F$ or $X = G$. A subspace of X is irreducible if it is so in the subspace topology. A topological space is called *noetherian* if every descending chain of closed subsets is eventually constant.

1. Prove that a topological space is irreducible if and only if every non-empty open subset in it is dense. Let X be an arbitrary topological space, $Y \subseteq X$. Verify that Y is irreducible if and only if $\overline{Y}$ is.

2. * Show that a noetherian topological space X can be expressed as a finite union

$$X = X_1 \cup \cdots \cup X_r \ ,$$

where the X_i are closed, irreducible, and none of them contains any other. Verify that this decomposition is unique up to the reordering of the terms. Show that $\mathbb{R}$ equipped with the finite complement topology is an irreducible noetherian space.

3. Let $X = [0,1]$, and consider the following collection τ of subsets:

$$\tau \stackrel{\text{def}}{=} \left\{ [0, \tfrac{1}{n}] \,|\, n \in \mathbb{N}^{>0} \right\} \cup \{\{0\}\} \cup \{\emptyset\} \ .$$

Prove that (X, τ) is an irreducible topological space. How long is the longest strictly descending chain of irreducible closed subsets?

4. The relation "$p \ominus q$ if for every discrete valued map d on X, $d(p) = d(q)$" is an equivalence relation, the equivalence classes of which are called *quasi-components.*
(i) Show that quasi-components are either equal or disjoint, and fill out X.
(ii) The quasi-components of a topological space are closed; each connected set is contained in a quasi-component.
(iii) Let $X \stackrel{\text{def}}{=} \{ \{ (0,0 \} , \{ (0,1) \} \} \cup \bigcup_{n=1}^{\infty} \{ \frac{1}{n} \} \times [0,1] \subseteq \mathbb{R}^2$. Then the points $(0,0)$ and $(0,1)$ are components, but not quasi-components.

5. Let $(X,\tau), (X,\sigma)$ two topologies on the same set, assume that $\sigma \subseteq \tau$. Does connectivity of a subset with respect to one topology imply anything for connectivity in the other?

6. Let C_n be an infinite sequence of connected subspaces of a topological space X, such that for every n, one has $C_n \cap C_{n+1} \neq \emptyset$. Show that $\bigcup_{n=1}^{\infty} C_n$ is connected as well.

7. Is it true that an infinite set is always connected in the finite complement topology?

8. * A topological space X is called *totally disconnected*, if the only connected sets are those consisting of one element only. Show that a discrete topological space is totally disconnected. Is the converse true?

9. Let $A \subseteq X$ be an arbitrary subspace, $C \subseteq X$ connected. Prove that $A \cap C \neq \emptyset$ and $(X - A) \cap C \neq \emptyset$ together imply $\partial A \cap C \neq \emptyset$.

10. Show that every convex subset $X \subseteq \mathbb{R}^n$ is connected.

11. (i) Prove that the spaces $[0,1]$, $[0,1)$, and $(0,1)$ are pairwise non-homeomorphic.
(ii) Show that $\mathbb{R} \not\approx \mathbb{R}^n$ for $n \geq 2$.

12. Verify that a continuous map $f : [0,1] \to [0,1]$ always has a fixed point; explain why the same statement is false for $[0,1)$.

13. Let (X,τ) be a topological space, $A \subseteq X$ a connected subspace. Is it true that $\text{int}_X A$ must be connected?