Topology (TOP) / Alex Küronya / Fall 2013

Homework 10

Due date: Dec 17th

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Prove that if $f : (X, x_0) \to (Y, y_0)$ is a homeomorphism, then

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

is an isomorphism.

2. ** Show that if X is a path-connected topological space, $h : X \to Y$ a continuous map, then h_* is independent of the base points chosen (up to appropriate isomorphisms). (Note: Your first task is to write it down precisely what it means for f_* to be independent of base points.)

3. * Prove that for any topological group G, the fundamental group $\pi_1(G, 1)$ is abelian.

4. Decide if the following statements are true:
   (1) If X is a discrete topological space with n elements (n a positive integer), $x_0 \in X$, then $\pi_1(X, x_0)$ is isomorphic to the cyclic group with n elements.
   (2) The fundamental group of a trivial topological space at an arbitrary base-point is trivial.

5. Let X be a path-connected topological space, $x_0, x_0' \in X$, h and h' two paths from x_0 to x_0'. Show that the $\tau_h = \tau_{h'} : \pi_1(X, x_0') \to \pi_1(X, x_0)$ whenever h is path-homotopic to h'.

6. ** Let $f_0, f_1 : (X, x_0) \to (Y, y_0)$ continuous maps homotopic via $F : X \times I \to Y$, denote

$$\begin{aligned} \gamma : I &\longrightarrow Y \\ t &\mapsto F(x_0, t) \ . \end{aligned}$$

Prove that

$$(f_1)_* = \tau_\gamma \circ (f_0)_* \ .$$

As a consequence show that $(f_1)_*$ is injective/surjective/trivial whenever $(f_0)_*$ is.

7. Let $f : X \to Y$ be a map, which is homotopic to a constant map. Show that f_* is the trivial homomorphism.

8. * (Homotopy invariance of the fundamental group) Let $f : (X, x_0) \to (Y, y_0)$ be a homotopy equivalence. Verify that $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.

9. Consider a covering map $p : E \to B$ with B connected. Prove that if for some $x \in B$ the fibre $p^{-1}(x) \subseteq E$ has m elements, then $p^{-1}(b)$ has m elements for every $b \in B$. (In this case the covering map p is called an *m-fold covering* of B.)

10. If $p : X \to Y$ and $q : Y \to Z$ are covering maps, and $(q \circ p)^{-1}(z)$ is finite for every $z \in Z$, then $q \circ p$ is also a covering map.

11. Let $p : E \to B$ be a covering map with E path-connected. Show that if B is simply connected, then p is a homeomorphism.

12. * Let $p : E \to B$ be a covering map. Show that if B is Hausdorff/regular, then so is E.

13. If X,Y are topological spaces, $x_0 \in X$, $y_0 \in Y$, then prove that
$$\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_0(Y, y_0) \ .$$

14. For a retract $A \subset \mathbb{D}^2$, prove that every continuous map $f : A \to A$ has a fixed point.

15. Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a continuous map such that $f(x) \neq f(-x)$ for every $x \in \mathbb{S}^2$. Show that g is surjective.

16. Determine the fundamental group of $\mathbb{S}^1 \times \mathbb{D}^2$ and of $\mathbb{S}^1 \times \mathbb{S}^2$.

17. Accepting the fact that for an arbitrary positive integer n, no antipode preseving map $f : \mathbb{S}^n \to \mathbb{S}^n$ is nullhomotopic, prove the following statements:
   (1) There exists no retraction $r : \mathbb{D}^{n+1} \to \mathbb{S}^n$.
   (2) There exists no antipode-preserving map $g : \mathbb{S}^{n+1} \to \mathbb{S}^n$.

18. Using the method for determining the fundamental group of the circle, prove that
$$\pi_1(T^2) \simeq \mathbb{Z} \times \mathbb{Z} \ .$$

19. Prove that a nonsingular 3×3 matrix with nonnegative real entries has a positive real eigenvalue.