Topology A (TOPA) / Alex Küronya / Fall 2009

Homework 6

Due date: November 10th

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. (Tube lemma) Let X be an arbitrary, Y a compact topological space, $x_0 \in X$ an arbitrary point, $N \subseteq X \times Y$ an open subset containing $\{ x_0 \} \times Y$. Prove that there exists an open neighbourhood of W of x_0 in X such that $N \supseteq W \times Y$.

2. Show that the product of two Hausdorff topological spaces is again Hausdorff.

3. * Prove or disprove that the product of two connected topological spaces is connected.

4. Decide whether the following is true: a function $f : Z \to X \times Y$ is continuous if and only it is continuous in both variables.

Definition. A *topological group* G is a topological space equipped with a group structure in such a way that

(1) the multiplication map $\mu : G \times G \to G$, $(g, h) \mapsto gh$ is continuous,

(2) taking inverse images $i : G \to G$, $g \mapsto g^1$ is continuous.

A *subgroup* of a topological group is an abstract subgroup with the subspace topology. If G and H are topological groups then a function $f : G \to H$ is a *homomorphism of topological groups*, if it is a homomorphism of abstract groups, which is continuous.

5. Let G be a group, which is a topological space at the same time. Show that G is a topological group iff the function $G \times G \to G$ sending (x, y) to xy^{-1} is continuous.

6. Verify that the following groups (equipped with the classical topology) are topological groups: $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}^+, \cdot)$, (complex numbers of absolute value 1, $\cdot$).

7. ** Prove that the general linear group $\mathrm{GL}(n, \mathbb{R})$ together with the usual matrix multiplication and forming inverse matrices is a topological group (here $\mathrm{GL}(n, \mathbb{R}) \subseteq \mathrm{Mat}_n(\mathbb{R})$ denotes the set of $n \times n$ invertible matrices; we give this group the topology inherited from $\mathrm{Mat}_n(\mathbb{R})$ thought of as $\mathbb{R}^{n^2}$).

8. ** Let G be a topological group, and denote the connected component of G containing the identity by G°. Show that G° is a closed normal subgroup of G.

9. We call a subset X in a topological group G *symmetric*, if $X^{-1} = X$. Show that the symmetric neighbourhoods of the identity element 1_G of G form a neighbourhood basis of 1_G.

10. ** Let X be a non-empty compact Hausdorff space with no isolated points. Show that X must be uncountable.

11. Prove that a connected metric space is either uncountable or has at most one point.

Definition. Let X be a topological space, $x \in X$. We say that X is *locally compact at* x, if there exists a compact subset $C \subseteq X$ which containts a neighbourhood of x. The space X is called *locally compact* if it is locally compact at every one of its points.

12. Prove that $\mathbb{R}^n$ is locally compact, but $\mathbb{Q}$ is not.