Topology A (TOPA) / Alex Küronya / Fall 2009

Homework 4

Due date: October 20th

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Find an example of a topological space which is not T_0, an example which is not T_1 but T_0, and one which is not T_2 but T_1.

2. Show that if X is a Hausdorff topological space, then a convergent sequence of points of X can have at most one limit point. Prove that a subspace of a Hausdorff space is itself Hausdorff with respect to the subspace topology.

3. * Verify that every connected open set in a locally path connected space is path connected.

4. Is it true that for every open subset $U \subseteq \mathbb{R}^n$, one has $U = \operatorname{int} \overline{U}$?

5. Let (X, d) be a metric space, $A \subseteq X$ an arbitrary fixed subset. For $x \in X$ define

$$d(x, A) \stackrel{\text{def}}{=} \inf_{y \in A} d(x, y) \ .$$

Show that the function $x \mapsto d(x, A)$ is continuous.

6. Let X be a second countable topological space, $A \subseteq X$ an uncountable set. Verify that uncountably many point of A are limit points of A.

7. ** Let R be a commutative ring (with 1), and let

$$\operatorname{Spec} R \stackrel{\text{def}}{=} \{\mathfrak{p} \subseteq R \,|\, \mathfrak{p} \text{ is a prime ideal of } R\} \ .$$

For an ideal $I \subseteq R$, consider the set

$$V(I) \stackrel{\text{def}}{=} \{\mathfrak{p} \in \operatorname{Spec} R \,|\, \mathfrak{p} \supseteq I\}$$

Show that the sets of the form $V(I)$ (where I runs through all ideals of R) satisfy the properties of closed sets of a topology. The topological space $\operatorname{Spec} R$ defined this way is called the *spectrum* of the ring R, the topology just defined on the set $\operatorname{Spec} R$ is the *Zariski topology*. How could you describe the Zariski topology on $R = \mathbb{C}[x]$? And on $\mathbb{Z}$?

8. ** Let $\phi : A \to B$ be a homomorphism of commutative rings (all rings in this course have 1).

(1) Prove that the function $\widetilde{\phi} : \operatorname{Spec} B \to \operatorname{Spec} A$ defined as

$$\widetilde{\phi}(P) \stackrel{\text{def}}{=} \phi^{-1}(P)$$

is a well-defined continuous function.

(2) If ϕ is a surjective homeomorphism, then $\widetilde{\phi}$ is a homeomorphism of $\operatorname{Spec} B$ onto the closed subset $V(\ker \phi) \subseteq \operatorname{Spec} A$.

(3) If ϕ is injective, then $\widetilde{\phi}(\operatorname{Spec} B) \subseteq \operatorname{Spec} A$ is a dense subset.

9. Let (X, τ) be a topological space, and consider the equivalence relation $x \sim y$ if for every $U \in \tau$, $x \in U$ if and only if $y \in U$. Is it true that $X/\sim$ is a T_0-space?

10. Let $f : X \to Y$ be an open continuous map. Prove that if X is first/second countable the so is $f(Y) \subseteq Y$.

11. Prove that if a topological space X has a countable dense subset, then every collection of disjoint open sets is countable.