PRACTICE SESSION # 6

1. Describe the following tensor products.
   (1) $\mathbb{Z}/(n) \otimes_{\mathbb{Z}} \mathbb{Z}/(m)$, where $m, n \in \mathbb{Z}$.
   (2) $k[x_1, \ldots, x_n] \otimes_k k[y_1, \ldots, y_m]$ for an arbitrary field k.
   (3) $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$.

2. Let V, W be finite-dimensional vector spaces over a field k. Show that there is a natural map
$$V^* \otimes_k W^* \longrightarrow (V \otimes_k W)^*$$
establishing an isomorphism between the two sides.

3. Consider the real vector space $V = \mathbb{R}^2$ with the standard basis $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Let $\phi, \psi : V \to V$ be linear maps given by the matrices
$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} -4 & -3 \\ -2 & -1 \end{pmatrix}$$
with respect to the ordered basis e_1, e_2. Compute the matrices of the linear maps $\phi \otimes \psi$ and $\psi \otimes \phi$ with respect to the ordered basis $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$.

4. Let $f : X \to Y$ be a morphism of affine varietes, $f^* : k[Y] \to k[X]$ the induced k-algebra homomorphism. Show that f^* is surjective if and only if f is injective.

5. Let $f : X \to Y$ be a regular map,
$$\Gamma_f \stackrel{\text{def}}{=} \{ (x, f(x) \mid x \in X \}$$
the *graph* of f. Prove that $\Gamma_f \subseteq X \times Y$ is a closed subvariety, and $\Gamma_f \simeq X$.

6. For affine varieties X and Y, the regular map $pr_Y : X \times Y \to Y$, $pr_Y(x, y) \stackrel{\text{def}}{=} y$ is called *projection to* Y.
   (1) If $Z \subseteq Y$ is a closed subset, $f : X \to Y$ a regular map, then show that
   $$f(Z) = pr_Y(\Gamma_f \cap Z) .$$
   (2) Prove that any regular map $f : X \to Y$ between affine algebraic sets factors as the composition of an inclusion and a projection. More precisely, show that for f as above, there exists a regular map $g : X \to X \times Y$ mapping X isomorphically onto a closed subset of $X \times Y$, and $f = pr_Y \circ g$.

7. Check that if $U \subseteq X$ is an open subset of a prevariety, then $(U, \mathcal{O}_X|_U)$ is a prevariety as well.

HOMEWORK

8. Let V be a finite-dimensional vector space $1 \leq i, j, k, l \leq n$ over an arbitrary field k, $v_1, \ldots, v_n$ a basis of V. Show that $\sum_{1 \leq i,j \leq n} \alpha_{ij} v_i \otimes v_j \in V \otimes V$ is an elementary tensor if and only if $\alpha_{ij}\alpha_{kl} = \alpha_{il}\alpha_{kj}$ for every combination of indices $1 \leq i, j, k, l \leq n$.

9. Show that the underlying topological space of a prevariety is noetherian.

10. Let $X, Y \subseteq \mathbb{A}^n$ be algebraic sets, and consider $\Delta \stackrel{\text{def}}{=} V(x_1 - y_1, \ldots, x_n - y_n) \subseteq \mathbb{A}^{2n}$ and $X \times Y \subseteq \mathbb{A}^{2n}$. Prove that the regular map

$$\phi : X \cap Y \to (X \times Y) \cap \Delta \ , \ x \mapsto (x, x)$$

is an isomorphism.

11. Consider the following collection of algebraic sets (all defined over the complex numbers). Which one of them are isomorphic? Justify your answer.

(1) $V(X^2 + Y^2 - 1)$
(2) $\mathbb{A}^1$
(3) $\mathbb{A}^2$
(4) $V(XY) \subseteq \mathbb{A}^2$
(5) $V(X^2 - Y^3)$
(6) $V(Y^2 - X^3 - X^2) \subseteq \mathbb{A}^2$
(7) $V(Y - X^2, Z - X^3) \subseteq \mathbb{A}^3$

12. Is it true that the image of an algebraic set under a regular map is again an algebraic set? How about the preimage?

13. The aim of this exercise is to define the field of rational functions on an arbitrary prevariety. Let X be a prevariety, (U, f) an ordered pair with $U \subseteq X$ open, and $f \in \mathcal{O}_X(U)$. Two such pairs (U, f) and (U', f') will be called equivalent (and denoted $(U, f) \sim (U', f')$), if there exists a non-empty open subset $V \subseteq U \cap U'$ of X for which

$$f|_U = f'|_{U'} \ .$$

(1) Verify that $\sim$ is indeed an equivalence relation.
(2) Explain with the difference from the equivalence relation we used to define the local ring of X at a point.
(3) Show that the equivalence classes of $\sim$ form a field. How can you define the field operations? This field is called the *field of rational functions of* X or the *function field of* X. It is denoted by $k(X)$.
(4) If X is an affine variety, then check that our new definition coincides with the old one.
(5) If $U \subseteq X$ is a non-empty open set, then $k(U) = k(X)$.

14. Let $Y \subseteq X$ be an irreducible closed subset of a prevariety $(X, \mathcal{O}_X)$. For $U \subseteq X$ we set $\mathcal{O}_Y(U)$ to be the ring of k-valued functions f on U such that for every point $x \in Y$ there exists an open neighbourhood V of x in X and an element $f \in \mathcal{O}_X(V)$ for which

$$F|_U = f \ .$$

Prove that $\mathcal{O}_Y$ with the pointwise restriction maps is a sheaf, and $(Y, \mathcal{O}_Y)$ is a prevariety.