PRACTICE SESSION # 5

1. For an arbitrary ring R, $R[[x]]$ denotes the ring of formal power series with coefficients in R. Verify the following.
   (1) $f = \sum_{k=0}^{\infty} a_k x^k$ is a unit if and only if $a_0 \in R$ is a unit.
   (2) If f is nilpotent then a_k is nilpotent for all $k \geq 0$.
   (3) ** If a_k is nilpotent for all $k \geq 0$, then f is nilpotent.

2. Let k be an arbitrary field, and let $k[[x_1, \ldots, x_n]]$ be the ring of formal power series in n variables and coefficients in k.
   (1) For $f \in (x_1, \ldots, x_n)$ verify that the formal expansion
   $$\frac{1}{1+f} = 1 - f + f^2 - f^2 + \ldots$$
   gives a well-defined element of $k[[x_1, \ldots, x_n]]$.
   (2) Show that $k[[x_1, \ldots, x_n]]$ is a local ring with $(x_1, \ldots, x_n)$ as its unique maximal ideal.

3. If $k = \mathbb{R}$ or $\mathbb{C}$, then one can define the ring $k\{x_1, \ldots, x_n\}$ of convergent power series with coefficients in k. Verify that $k\{x_1, \ldots, x_n\}$ is a local ring with $(x_1, \ldots, x_n)$ as its unique maximal ideal.

4. Let $\mathcal{F}$ be a sheaf of rings on the topological space X, $x \in X$. Check that the stalk $\mathcal{F}_x$ is indeed a ring.

5. Let $\phi : \mathcal{F} \to \mathcal{G}$ be an isomorphism of sheaves of rings on X, $x \in X$. Verify that the induced morphism $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is again an isomorphism.

6. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. We define that *presheaf kernel* and the *presheaf image* of ϕ by $U \mapsto \ker \phi(U)$, and $U \mapsto \operatorname{im} \phi(U)$, respectively. Show that the presheaf kernel of ϕ is a sheaf, but the presheaf image is only a presheaf.

HOMEWORK

7. Let $r \in R$ be a nilpotent element. Prove that $r + 1 \in R$ is a unit. Generalizing this, show that the sum of a nilpotent element and a unit is a unit.

8. Consider an arbitrary ring R, and the ring of polynomials $R[x]$ with coefficients in R, let
$$f = a_n x_n + \ldots a_1 x + a_0 \in R[x] .$$
Verify the following claims.
   (1) f is a unit in $R[x]$ if and only if $a_0 \in R^{\times}$ and a_i is nilpotent for $i \geq 1$.
   (2) f is nilpotent precisely if all of its coefficients are nilpotent.
   (3) f is a zero-divisor if and only if there exists a non-zero element $r \in R$ such that $rf = 0$.

9. Consider the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{ha } x \geq 0 \\ 0 & \text{ha } x \leq 0 \ . \end{cases}$$

Show that $f \in \mathcal{C}^\infty(\mathbb{R})$, and compute its Taylor series at $x = 0$. Determine the set of points in which the Taylor series converges to f.

10. For a morphism of sheaves $\phi : \mathcal{F} \to G$ and a point $x \in X$, prove that

$$\ker(\phi_x) = (\ker \phi)_x \ .$$

11. Let X be a variety over an algebraically closed field k, Y a closed subset of X. We define $\mathcal{I}_{Y/X}$ the *sheaf of ideals of* Y *in* X as follows. For $U \subseteq X$ open, set

$$\mathcal{I}_{Y/X}(U) \stackrel{\text{def}}{=} \text{regular functions on } U\text{, which vanish identically on } U \cap Y \ .$$

Prove that $\mathcal{I}_{X/Y}$ is indeed a sheaf, and it is a subsheaf of $\mathcal{O}_X$.

Definition. Let R be an integral domain, M an R-module, $x \in M$. We call x a *torsion element* of M, if there exists $r \neq 0$ in R for which $rx = 0$. The set of torsion elements of M is called the *torsion submodule* of M, and denoted by $T(M)$. If $T(M) = 0$, then M is *torsion-free.*

12. Prove the following claims about torsion submodules.

(1) $T(M)$ is indeed a submodule of M.
(2) The quotient module $M/T(M)$ is torsion-free.
(3) If $f : M_1 \to M_2$ is an R-module homomorphism, then $f(T(M_1)) \subseteq T(M_2)$.
(4) If the sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is exact, then so is

$$0 \longrightarrow T(M') \longrightarrow T(M) \longrightarrow T(M'') \longrightarrow 0 \ .$$

(5) If $S \subseteq R$ is a multiplicative subset, then $S^{-1}(T(M)) = T(S^{-1}(M))$.
(6) Being torsion-free is a local property (more precisely, show that M is torsion-free $\Leftrightarrow$ M_P is torsion-free for all prime ideals $P \subseteq R \Leftrightarrow M_{\mathfrak{m}}$ is torsion-free for all maximal ideals $\mathfrak{m} \subseteq R$).