Algebraic combinatorics / Fall 2009 / Alex Küronya

Practice Session # 3.

1. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ be a partition, m a positive integer. The generalized elementary/complete symmetric polynomials in m variables are defined as follows.

$$e_\lambda(X) \stackrel{\text{def}}{=} e_{\lambda_1}(X) \cdot \dots \cdot e_{\lambda_k}(X) \ ,$$
$$h_\lambda(X) \stackrel{\text{def}}{=} h_{\lambda_1}(X) \cdot \dots \cdot h_{\lambda_k}(X) \ .$$

Check that e_λ and h_λ are indeed symmetric polynomials. Show that the set

$$\{e_\lambda(X) \mid \lambda \text{ is a partition }\}$$

is a system of free generators of the ring of symmetric polynomials in m variables over R.

2. Verify the following identities in the polynomial ring $R[X_1, \dots, X_m, T]$.

$$\prod_{i=1}^{m} \frac{1}{1 - X_i T} = \sum_{n \geq 0} h_n(X) T^n$$
$$\sum_{k=0}^{n} (-1)^k e_k(X) h_{n-k}(X) = 0 \ .$$

Using these results show that

$$R[h_1(X), \dots, h_n(X)] = R[e_1(X), \dots, e_n(X)] \ .$$

Prove that the complete symmetric polynomials generate the ring of symmetric polynomials over R.

3. Let $p_n(X_1, \dots, X_m) \stackrel{\text{def}}{=} X_1^n + \dots + X_m^n$ be the n^{th} power sum. Check the relations below.

$$n e_n(X) - p_1(X) e_{n-1}(X) + p_2(X) e_{n-2}(X) - \dots + (-1)^n p_n(X) = 0$$
$$n h_n(X) - p_1(X) h_{n-1}(X) - p_2(X) h_{n-2}(X) - \dots - p_n(X) = 0$$

Definition. Let $\rho : G \to \mathrm{GL}(V)$ be a representation of the finite group G. A subspace $W \subseteq V$ is *invariant*, if

$$\rho(g)(W) \subseteq W \ .$$

holds for every $g \in G$. The representation ρ is called *irreducible*, if V has no non-trivial invariant subspaces.

4. Show that all irreducible representations of an abelian group G are one-dimensional.

5. Verify the following identities for polynomials in three variables.

$$e_2(X) = \begin{vmatrix} h_1(X) & h_2(X) \\ h_0(X) & h_1(X) \end{vmatrix} \ , \ e_3(X) = \begin{vmatrix} h_1(X) & h_2(X) & h_3(X) \\ h_0(X) & h_1(X) & h_2(X) \\ 0 & h_0(X) & h_1(X) \end{vmatrix}$$

Definition. The *content* $\mu = (\mu_1, \mu_2, \dots)$ of a Young tableau T is the sequence of natural numbers for which the number of 1's in T is μ_1, the number of 2's in T is μ_2, and so on.

If λ, μ are partitions, then the *Kostka number* $K_{\lambda,\mu}$ is the number of Young tableaux T with shape λ and content μ.

6. Let λ, μ be partitions. Prove the following statements.
   (1) If $|\lambda| \neq |\mu|$ then $K_{\lambda,\mu} = 0$.
   (2) $K_{\lambda,\lambda} = 1$.
   (3) If $\mu \not\leq \lambda$ with respect to the lexicographic order[1], then $K_{\lambda,\mu} = 0$.
   (4) $K_{\lambda,\mu} \neq 0$ exactly if
   $$\mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i$$
   holds for every $i \geq 1$.
   (5) Calculate $K_{\lambda,\mu}$ when $\lambda = (3,2)$ and $\mu = (1,1,1,1,1)$.

Homework

7. Prove the relation below for polynomials in three variables.
$$s_{(2,1)(X)} = \begin{vmatrix} h_2(X) & h_3(X) \\ h_1(X) & h_1(X) \end{vmatrix}$$

8. The tableau below is the result of a row insertion. If the new box is the circled one, what was the original tableau and the element we inserted?

1	2	2	3	5
2	3	6	6	
4	4	7	7	
5	6			

9. ** Verify the Jacobi–Trudi formula for an arbitrary partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$.
$$s_\lambda(X_1, \dots, X_m) = \det(h_{\lambda_i+j-i}(X))_{1\leq i,j\leq k} .$$

10. ** Prove the following form of the Jacobi–Trudi identity (this was the original definition of Schur polynomials).
$$s_\lambda(x_1, \dots, x_m) = \frac{\det(X_j^{\lambda_i+m-j})_{1\leq i,j\leq m}}{\det(X_j^{m-i})_{1\leq i,j\leq m}} .$$

[1] $\mu \leq \lambda$ in the lexicographic order if $\mu_i < \lambda_i$ for the first index where $\mu_i \neq \lambda_i$.