Algebraic combinatorics / Fall 2009 / Alex Küronya

Problem Session #2

1. Determine the Schur polynomials in three variables belonging to the partitions $(2,1), (2,1,1)$.

2. Add the element 2 to the Young tableau below using row insertion.

1	2	2	3	5
2	3	6	6	
4	4	7	7	
5	6			

3. Prove that whenever we are given a Young tableau with the location of the new box, we can recover the tableau before the row-insertion and the value we had added.

4. Multiply the following two tableaux via iterated row insertion.

1	2	2	3
3	4		
5	5		

1	3
2	

5. Let G be a finite group, V a finite-dimensional vector space (over an arbitrary field), $\rho : G \to GL(V)$ a representation of G. We define the dual representation of ρ via

$$\rho^* : G \to GL(V^*) \ ,$$
$$\rho^*(g) = \rho(g^{-1})^T \ .$$

Verify that ρ^* is indeed a representation of G. Furthermore, if $\langle f^*, e\rangle = f^*(e)$ with respect to some basis, then

$$\langle \rho^*(g)(v^*), \rho(g)(v)\rangle = \langle v^*, v\rangle \ .$$

6. Prove that the functions below are representations of the symmetric group S_3.

(1) $\alpha : S_3 \to GL(\mathbb{C})$, $\alpha(g)(v) \stackrel{\text{def}}{=} \text{sgn}(g)v$.

(2) If x_1, x_2, x_3 are coordinates on $\mathbb{C}^3$, then

$$\beta(g)((x_1, x_2, x_3)) = (x_{g^{-1}(1)}, x_{g^{-1}(2)}, x_{g^{-1}(3)}) \ .$$

In the latter case find the maximal invariant subspace. Does $\mathbb{C}^3$ have a subspace with the property that β restricted to that subspace is irreducible?

7. Add the element 3 to the tableau via row insertion.

1	1	1	1	5
2	2	6	6	
4	5	7	7	
5	8			

8. Calculate the Schur polynomial in four variables associated to the partition $(2, 2)$.

9. (Schur lemma) Let V, W be irreducible representations of the finite group G, $\phi : V \to W$ a G-module homomorphism. Then

(1) Either $\phi = 0$ or ϕ is an isomorphism;
(2) if $V = W$ and $\phi \neq 0$, then $\phi = \lambda I_V$ for some $\lambda \in \mathbb{C}$.

10. Compute the product of the two tableaux using iterated row insertion.

1	5	5	5
3	6		
7	8		

1	2
5	