Algebraic combinatorics / Fall 2009 / Küronya Alex

Problem Session #1

Let $R$ be an arbitrary ring (commutative with an identity). We call a polynomial $f \in R[X]$ *formally of degree* $n$, if $\deg f \leq n$. The *resultant* of the polynomials $f$ and $g$ of formal degree $m$ and $n$, respectively, is defined as follows: if

$$f(X) \;=\; a_m X^m + \cdots + a_1 X + a_0 \;\; \text{és} \; g(X) \;=\; b_n X^n + \cdots + b_1 X + b_0 \;,$$

then

$$\operatorname{res}(f,g) \stackrel{\text{def}}{=} \left| \begin{array}{ccccccccc} a_m & a_{m-1} & \cdot & \cdot & a_0 & & & & \\ & a_m & a_{m-1} & \cdot & \cdot & a_0 & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & & a_m & a_{m-1} & \cdot & \cdot & a_0 \\ & & & & & a_m & a_{m-1} & \cdot \;\; \cdot & a_0 \\ b_n & b_{n-1} & \cdot & \cdot & b_0 & & & & \\ & b_n & b_{n-1} & \cdot & \cdot & b_0 & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & & b_n & b_{n-1} & \cdot & \cdot & b_0 \\ & & & & & b_n & b_{n-1} & \cdot \;\; \cdot & b_0 \end{array} \right| .$$

All elements which are not named or represented by a dot are considered to be zero.

If $f \in R[X]$ is a non-constant monic polynomial of degree $n$, then the *discriminant* of $f$ is given by the following formula:

$$\Delta_f \stackrel{\text{def}}{=} (-1)^{n(n-1)/2} \cdot \operatorname{res}(f, f') \;,$$

where $f'$ denotes the derivative of $f$.

1. Verify the following properties of resultants.
   (1) If $r, s \in R$ then $\operatorname{res}(rf, sg) = r^n s^m \cdot \operatorname{res}(f,g)$.
   (2) $\operatorname{res}(g,f) = (-1)^{mn} \operatorname{res}(f,g)$.
   (3) Let $\phi : R \to S$ be a ring homomorphism; we will denote by $\phi \circ f$ and $\phi \circ g$ the images of $f$ and $g$ in $S[X]$. Then

$$\operatorname{res}(\phi \circ f, \phi \circ g) \;=\; \phi \circ \operatorname{res}(f,g) \;.$$

2. Prove the following classical results on discriminants of polynomials.
   (1) If $f = X^2 + pX + q$ then $\Delta_f = p^2 - 4q$.
   (2) If $f = X^3 + pX^2 + qX + r$ then $\Delta_f = p^2q^2 + 18pqr - 4p^3r - 4q^3 - 27r^2$.
   (3) If $f = X^m + aX + b$ and $m \geq 2$ then $\Delta_f = (-1)^{m(m-1)/2}((1-m)^{m-1}a^m + m^m b^{m-1})$.

3. Write the symmetric polynomial $X_1^3 + X_2^3 + X_3^3 \in R[X_1, X_2, X_3]$ as a polynomial in the elementary symmetric polynomials in three variables.

4. Consider the polynomial ring $R[X]$, and let

$$h \;=\; a_n X^n + a_{n-1} X^{n-1} + \cdots + a_0 \;\; \in \; R[X]$$

be such that $a_n \in R^\times$.
(*i*) Think it through whether one has division with remainder for $h$ in $R[X]$.
(*ii*) Show that every $f \in R[X]$ can be written uniquely in the form

$$f \;=\; f_{n-1} X^{n-1} + \cdots + f_1 X + f_0$$

with $f_i \in R[h]$. Furthermore, every $f_i$ has a unique expression as

$$f_i \;=\; \sum_{j \geq 0} a_{ij} h^j \;,\; a_{ij} \in R \;.$$

$(iii)$ Prove that $h$ is algebraically independent over $R$, and $\{1, X, X^2, \ldots, X^{n-1}\}$ is a free generating system of $R[X]$ as an $R[h]$-module.

## Homework

5. Compute the following resultants (where $m$ and $n$ denote the formal degree of $f$ and $g$, respectively).

(1) $m = 1$, $g \in R$.
(2) $m = n = 1$, $f = a_1X + a_0$, $g = b_1X + b_0$.
(3) $f = a_2X^2 + a_1X + a_0$ , $g = b_2X^2 + b_1X + b_0$.

6. Verify that whenever $f, g \in R[X]$ are monic polynomials, we always have

$$\Delta_{fg} = \Delta_f \cdot \Delta_g \cdot \mathrm{res}(f, g)^2 .$$

7. Decide whether the following claim holds: if $f \in R[X]$ is a monic polynomial, $c \in R$, then

$$\Delta_f = \Delta_{f(X+c)} .$$

8. Let $M$ be a free $R$-module (that is, assume that $M$ has a free system of generators over $R$). Prove that any free generating system of $M$ over $R$ has the same cardinality. This common cardinality is called the *rank* of $M$ over $R$.

9. Let $k$ be a field, $\phi \in k(X_1, \ldots, X_n)$ a symmetric rational function. Show that $\phi \in k(s_1, \ldots, s_n)$, that is, that $\phi$ can be expressed as a rational function of the elementary symmetric polynomials.

10. Let $R \subseteq S$,$S \subseteq T$ be ring extensions, $\mathcal{G}_1 \subseteq S$ a free generating system of $S$ over $R$, $G_2 \subseteq T$ a free system of generators of $T$ over $S$. Show that

$$\mathcal{G}_1\mathcal{G}_2 \overset{\text{def}}{=} \{g_1g_2 \mid g_1 \in \mathcal{G}_1 \, , \, g_2 \in \mathcal{G}_2\}$$

is a free generating system of $T$ over $R$.

11. For a non-zero polynomial $f \in k[X]$, $\alpha \in k$ is a multiple zero of $f$ exactly if $f'(\alpha) = 0$.