Topology A (TOPA) / Alex Küronya / Fall 2008

Homework 8

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Show that a retract of a contractible topological space is contractible.

2. ** Prove that for any pair of maps $f, g : \mathbb{S}^n \to \mathbb{S}^n$ with $\forall x \in \mathbb{S}^n\ f(x) \neq -g(x)$, one has $f \simeq g$.

3. Let $f : X \to Y$ be homotopic maps, with f being a homeomorphism. Does it follow that g is a homeomorphism as well? Prove your answer.

4. Which of the following topological properties are preserved under homotopy equivalence of topological spaces (i.e. if $X \simeq Y$ and X has the property in question then does Y have it automatically?): compactness, connectedness, path-connectedness, first countability, second countability, T_0, T_1, T_2, T_3, T_4?

5. Which of the following properties are preserved under homotopy of maps: injective, surjective, open, closed, perfect, proper?

6. Let X,Y be topological spaces, Y discrete. Verify that the projection map $\pi : X \times Y \to X$ is a covering map.

7. * Consider a covering map $p : E \to B$ with B connected. Prove that if for some $x \in B$ the fibre $p^{-1}(x) \subseteq E$ has m elements, then $p^{-1}(b)$ has m elements for every $b \in B$. (In this case the covering map p is called an *m-fold covering* of B.)

8. ** Let $\phi : G \to H$ be a map of topological groups (that is, ϕ should be a continuous homomorphism).
(i) Verify that $\ker \phi \subseteq G$ is a closed normal subgroup of G.
(ii) Show that if G is compact and ϕ surjective, then

$$G/\ker \phi \approx H$$

as topological groups (in other words: prove that the map $\widetilde{\phi} : G/\ker \phi \to H$ induced by ϕ is a homeomorphism of topological spaces *and* an isomorphism of groups).

Definition A subspace $A \subseteq X$ is a *strong deformation retract of* X (or: X *strongly deformation retracts onto* A) if there exists a homotopy of maps $F : X \times I \to X$ such that
(1) for every $x \in X$ one has $F(x, 0) = x$;
(2) for every $x \in X$ one has $F(x, 1) \in A$;
(3) for every $a \in A$, and every $t \in I$ one has $F(a, t) = a$.

9. * Show that if a topological space X strongly deformation retracts to a point $x \in X$, then for each open neighbourhood U of X there exists an open neighbourhood $x \in V \subseteq U$ such that the inclusion map $V \hookrightarrow U$ is nullhomotopic.