Topology (TOP) / Alex Küronya / Fall 2006

Homework 9

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. * Show that every covering map $p : E \to B$ is open.

2. * Let $p : E \to B$ and $p' : E' \to B'$ be covering spaces. Verify that $p \times p' : E \times E' \to B \times B'$ is a covering space as well.

3. ** Let $p : E \to B$ be a covering map, $p(e_0) = b_0.$, $F : I \times I \to B$ a continuous map with $F(0,0) = b_0$. Prove that there exists a unique lifting $\tilde{F} : I \times I \to E$ such that $\tilde{F}(0,0) = e_0$. Show that if in addition F is a path homotopy, then so is $\tilde{F}$.

4. (Functorial properties of f_*) Let $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$ be continuous maps. Show that

$$(g \circ f)_* = g_* \circ f_* \ ,$$

moreover, verify that if i is the identity map, that it induces the identity homomorphism on the fundamental groups.

5. Prove that if $f : (X, x_0) \to (Y, y_0)$ is a homeomorphism, then

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

is an isomorphism.

6. ** For any topological group G, the fundamental group $\pi_1(X, 1)$ is abelian.

7. If $p : X \to Y$ and $q : Y \to Z$ are covering maps, and $(q \circ p)^{-1}(z)$ is finite for every $z \in Z$, then $q \circ p$ is also a covering map.

8. Consider a covering map $p : E \to B$ with B connected. Prove that if for some $x \in B$ the set $p^{-1}(x)$ has m elements (m a positive integer), then $p^{-1}(b)$ has m elements for every $b \in B$.

9. Let $p : E \to B$ be a covering map. Show that if B is Hausdorff/regular, then so is E.