Topology (TOP) / Alex Küronya / Fall 2006

Homework 7

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Let X, Y be topological space, $A \subseteq X$ a closed subset, $f : A \to Y$ a map. Consider the quotient space $Y \cup_f X$. Show that the natural inclusion $Y \hookrightarrow Y \cup_f X$ maps Y onto a closed subspace, while the image of the inclusion $X - A \hookrightarrow Y \cup_f X$ is open.

2. * Prove that if $f : X \to Y$, $g : Z \to W$ are open identification maps, then so is $f \times g : X \times Z \to Y \times W$, where $(f \times g)(x, z) \stackrel{\text{def}}{=} (f(x), g(z))$.

3. Let X, Y be normal topological spaces, $A \subseteq X$ a closed subset with a map $f : A \to Y$. Verify that $Y \cup_f X$ is normal as well.

4. ** Let G be a topological group, $H \subseteq G$ a closed subgroup. Prove that the space

$$G/H \stackrel{\text{def}}{=} \{ gH \mid g \in G \}$$

of left cosets of H equipped with the quotient topology induced by the canonical projection $\pi : G \to G/H$ is a Hausdorff topological space. Verify in addition that the map π is open.

5. Verify the following claims for an arbitrary topological group G:
   (1) The diagonal map $\delta : G \to G \times G$, $\delta(x) \stackrel{\text{def}}{=} (x, x)$ is closed.
   (2) $\{ 1 \} \subseteq G$ is a closed subgroup.
   (3) The intersection of all neighbourhoods of 1 is $\{ 1 \}$.

6. * Let G be a topological group, H a subgroup of G. Prove that G/H is a Hausdorff topological space if and only if $H \subseteq G$ is a closed subgroup.

7. For a topological group G, show that every open subgroup of G is closed, and every closed subgroup of finite index in G is open.

8. Prove that if G is a topological group, H a subgroup of G, then G/H is discrete if and only if $H \subseteq G$ is open.

9. Show that the product of two quotient maps need not be a quotient map.